Funções Hiperbólicas

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- Análogas de muitas formas às funções trigonométricas;
- Relacionam-se com as hipérboles, ao passo que as funções trigonométricas relacionam-se com o círculo.

Funções Hiperbólicas Básicas

Cosseno Hiperbólico: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Seno Hiperbólico: $\operatorname{senh} x = \dfrac{e^x - e^{-x}}{2}$

Tangente Hiperbólico: $\operatorname{tgh} x = \dfrac{\operatorname{senh} x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

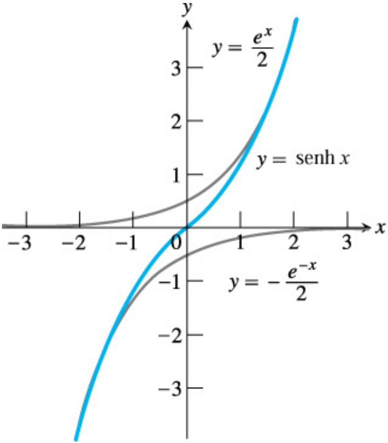
Cotangente Hiperbólico: $\operatorname{cotgh} x = \dfrac{\cosh x}{\operatorname{senh} x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Secante Hiperbólica: $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

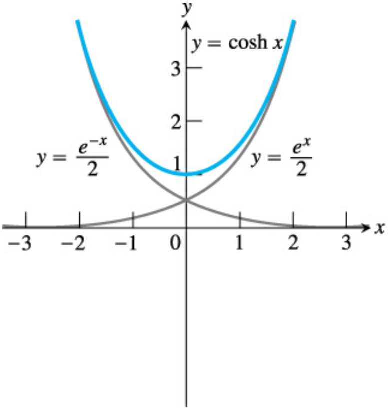
Cossecante Hiperbólica: $\operatorname{cossech} x = \dfrac{1}{\operatorname{senh} x} = \dfrac{2}{e^x - e^{-x}}$

Identidades

$$
\begin{aligned}
\operatorname{senh}(-x) &= -\operatorname{senh} x \\
\cosh(-x) &= \cosh(x) \\
\operatorname{senh} 2x &= 2\operatorname{senh} x \cosh x \\
\cosh 2x &= \cosh^2 x + \operatorname{senh}^2 x \\
\operatorname{senh}(x+y) &= \operatorname{senh} x \cosh y + \cosh x \operatorname{senh} y \\
\cosh(x+y) &= \cosh x \cosh y + \operatorname{senh} x \operatorname{senh} x \\
\cosh^2 x &= \frac{\cosh 2x + 1}{2} \\
\operatorname{senh}^2 x &= \frac{\cosh 2x - 1}{2} \\
\cosh^2 x - \operatorname{senh}^2 x &= 1 \\
\operatorname{tgh}^2 x &= 1 - \operatorname{sech}^2 x \\
\operatorname{cotgh}^2 x &= 1 + \operatorname{cossech}^2 x
\end{aligned}
$$

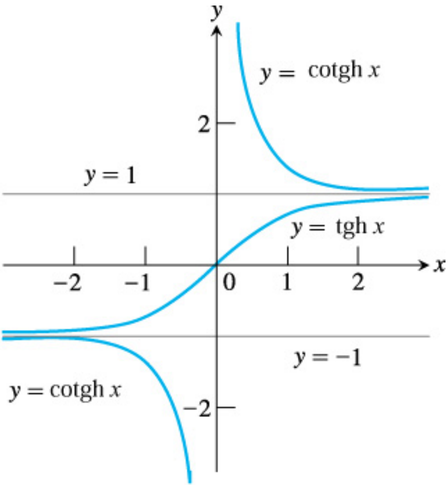


(a) O seno hiperbólico e suas componentes exponenciais

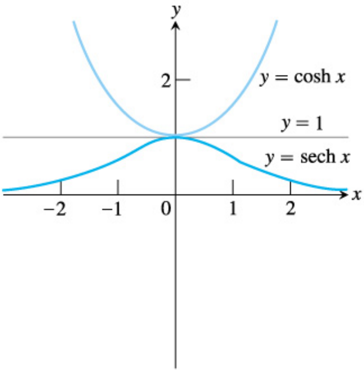


(b) O cosseno hiperbólico e suas componentes exponenciais

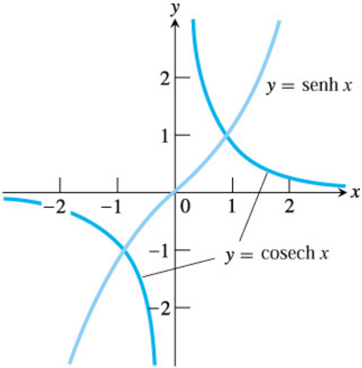
Gráfico de Funções Hiperbólicas
Funções Tangente e Cotangente Hiperbólicos



(c) Os gráficos de $y = \text{tgh}\, x$ e $y = \text{cotgh}\, x = 1/\text{tgh}\, x$



(d) Os gráficos de $y = \cosh x$ e $y = \operatorname{sech} x = 1/\cosh x$



(e) Os gráficos de $y = \operatorname{senh} x$ e $y = \operatorname{cosech} x = 1/\operatorname{senh} x$

- **Função Seno:**

$$\frac{d}{dx}(\operatorname{senh} x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

- **Função Cosseno:**

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \operatorname{senh} x$$

Tabela de Derivadas!

$$\begin{aligned}
\frac{d}{dx}(\operatorname{senh} x) &= \cosh x \\
\frac{d}{dx}(\cosh x) &= \operatorname{senh} x \\
\frac{d}{dx}(\operatorname{tgh} x) &= \operatorname{sech}^2 x \\
\frac{d}{dx}(\operatorname{cossech} x) &= -\operatorname{cossech} x \, \operatorname{cotgh} x \\
\frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \, \operatorname{tgh} x \\
\frac{d}{dx}(\operatorname{cotgh} x) &= -\operatorname{cossech}^2 x
\end{aligned}$$

Tabelas de Derivadas!

Sabendo que:

- Cosseno Hiperbólico: $\cosh x = \dfrac{e^x + e^{-x}}{2}$
- Seno Hiperbólico: $\operatorname{senh} x = \dfrac{e^x - e^{-x}}{2}$

Demonstre os resultados da Tabela de Derivadas de Funções Hiperbólicas!

Tabela de Derivadas — Regra da Cadeia !

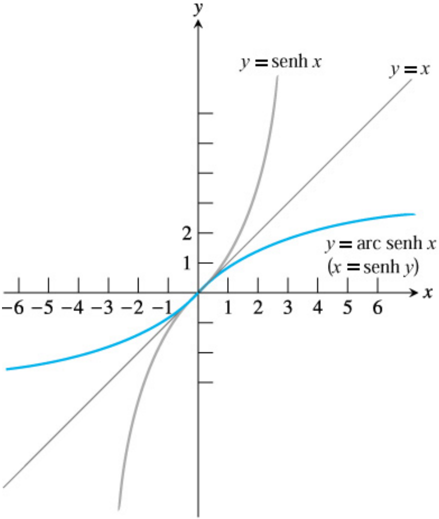
$$
\begin{aligned}
\frac{d}{dx}(\operatorname{senh} u(x)) &= \cosh u(x)\,\frac{d}{dx}(u(x)) \\
\frac{d}{dx}(\cosh u(x)) &= \operatorname{senh} u(x)\,\frac{d}{dx}(u(x)) \\
\frac{d}{dx}(\operatorname{tgh} u(x)) &= \operatorname{sech}^2 u(x)\,\frac{d}{dx}(u(x)) \\
\frac{d}{dx}(\operatorname{cossech} u(x)) &= -\operatorname{cossech} u(x)\,\operatorname{cotgh} u(x)\,\frac{d}{dx}(u(x)) \\
\frac{d}{dx}(\operatorname{sech} u(x)) &= -\operatorname{sech} u(x)\,\operatorname{tgh} u(x)\,\frac{d}{dx}(u(x)) \\
\frac{d}{dx}(\operatorname{cotgh} u(x)) &= -\operatorname{cossech}^2 u(x)\,\frac{d}{dx}(u(x))
\end{aligned}
$$

Funções Hiperbólicas Inversas

$$
\begin{aligned}
y &= \operatorname{arc\,senh} x &\iff& \operatorname{senh} y = x \\
y &= \operatorname{arc\,cosh} x &\iff& \cosh y = x \\
y &= \operatorname{arc\,tgh} x &\iff& \operatorname{tgh} y = x \\
y &= \operatorname{arc\,cossech} x &\iff& \operatorname{cossech} y = x \\
y &= \operatorname{arc\,sech} x &\iff& \operatorname{sech} y = x \\
y &= \operatorname{arc\,cotgh} x &\iff& \operatorname{cotgh} y = x
\end{aligned}
$$

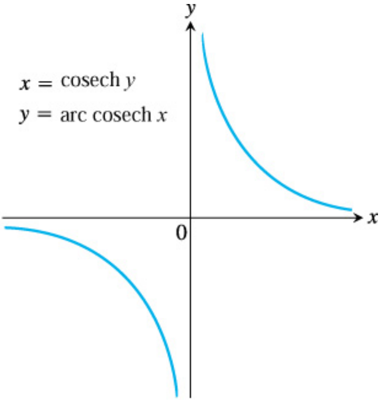
Domínio e imagem

Estude o domínio e a imagem das funções hiperbólicas inversas



y
y = senh x
y = x
y = arc senh x
(x = senh y)
2
1
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6
x

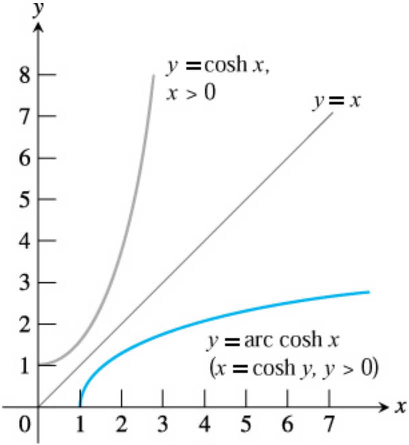
(a)



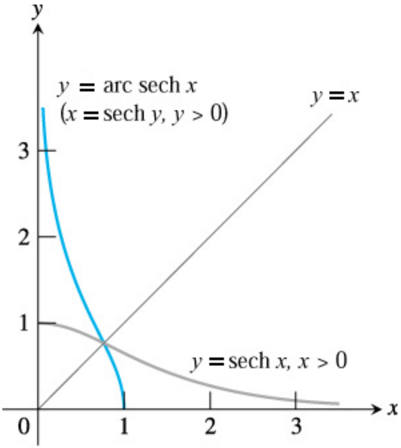
y
x = cosech y
y = arc cosech x
0
x

(c)

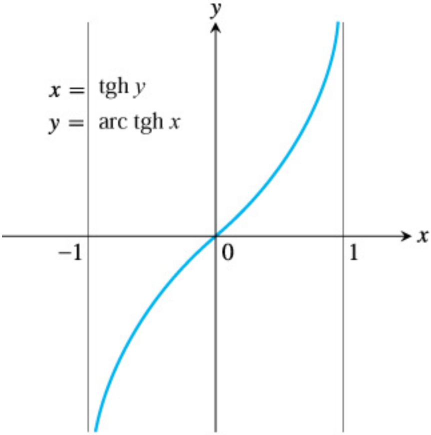
Gráfico Funções Hiperbólicas Inversas
Funções Hiperbólicas Inversas de Cosseno e Secante



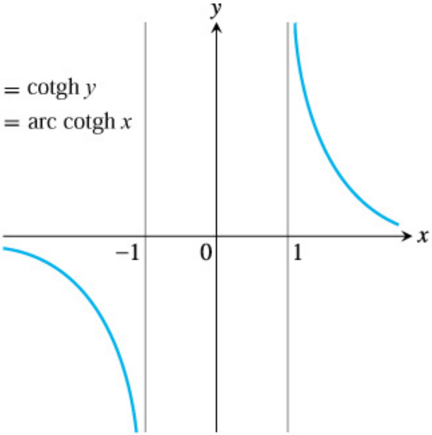
(b)



(c)



(a)



(b)

Tabela de Derivadas!

$$
\begin{aligned}
\frac{d}{dx}\left(\operatorname{arc\,senh} u(x)\right) &= \frac{1}{\sqrt{1+u^2}}\frac{d}{dx}u(x) & & \\
\frac{d}{dx}\left(\operatorname{arc\,cosh} u(x)\right) &= \frac{1}{\sqrt{u^2-1}}\frac{d}{dx}u(x) & & u(x) > 1 \\
\frac{d}{dx}\left(\operatorname{arc\,tgh} u(x)\right) &= \frac{1}{1-u^2}\frac{d}{dx}u(x) & & |u(x)| < 1 \\
\frac{d}{dx}\left(\operatorname{arc\,cossech} u(x)\right) &= -\frac{1}{|u|\sqrt{u^2+1}}\frac{d}{dx}u(x) & & u(x) \neq 0 \\
\frac{d}{dx}\left(\operatorname{arc\,sech} u(x)\right) &= -\frac{1}{u\sqrt{1-u^2}}\frac{d}{dx}u(x) & & 0 < u(x) < 1 \\
\frac{d}{dx}\left(\operatorname{arc\,cotgh} u(x)\right) &= \frac{1}{1-u^2}\frac{d}{dx}u(x) & & |u(x)| > 1
\end{aligned}
$$

As funções hiperbólicas inversas são diferenciáveis porque as funções hiperbólicas são diferenciáveis.

Segundo Trabalho!

Demonstre os resultados da Tabela de Derivada de Funções Hiperbólicas Inversas!

Aviso para o Primeiro e Segundo Trabalhos!

- Data de entrega: 30 de outubro de 2006 – segunda-feira!
- Grupo: **máximo** 3 pessoas!
- Horário: os trabalhos serão recebidos até às 21h.

Tabela de Integrais das Funções Hiperbólicas

$$
\begin{aligned}
\int \operatorname{senh} u \, du &= \cosh u + C \\
\int \cosh u \, du &= \operatorname{senh} u + C \\
\int \operatorname{sech}^2 u \, du &= \operatorname{tgh} u + C \\
\int \operatorname{cossech}^2 u \, du &= -\operatorname{cotgh} u + C \\
\int \operatorname{sech} u \ \operatorname{tgh} u \, du &= -\operatorname{sech} u + C \\
\int \operatorname{cossech} u \ \operatorname{cotgh} u \, du &= -\operatorname{cossech} u + C
\end{aligned}
$$

Integrais que conduzem a Funções Hiperbólicas Inversas
Definição

Tabela

$$\int \frac{du}{\sqrt{a^2+u^2}} = \text{arc senh}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \text{arc cosh}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\,\text{arc tgh}\,\dfrac{u}{a} + C, & \text{se } u^2 < a^2 \\ \dfrac{1}{a}\,\text{arc cotgh}\,\dfrac{u}{a} + C, & \text{se } u^2 > a^2 \end{cases}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\,\text{arc sech}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\,\text{arc cossech}\left|\frac{u}{a}\right| + C, \qquad a \neq 0$$