An Introduction to Nonlinearity

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An Introduction to Nonlinearity

- Introduction
- Historical perspective
- Nonlinear stiffness
  - Symmetry
  - Asymmetry
- Nonlinear damping
- Conclusions
Dynamical System

Inertia force

+ 

Damping force

+ 

Stiffness force

= 

Excitation force
Common stiffness nonlinearities

- Hardening
- Softening
- Cubic Stiffness
- Bilinear Stiffness
- Saturation (or limiter)
- Clearance (or backlash)
Common damping nonlinearities

Coulomb Friction

Nonlinear Damping
Historical Perspective
Historical Perspective

Galileo Galilei, 1564–1642, studied the pendulum. He noticed that the natural frequency of oscillation was roughly independent of the amplitude of oscillation.

Christiaan Huygens, 1629–1695, patented the pendulum clock in 1657. He discovered that wide swings made the pendulum inaccurate because he observed that the natural period was dependent on the amplitude of motion, i.e. it was a nonlinear system.
Robert Hooke, 1635–1703, is famous for his law which gives the linear relationship between the applied force and resulting displacement of a linear spring.

Isaac Newton, 1643–1727, is of course, famous for his three laws of motion.
Historical Perspective

**John Bernoulli**, 1667–1748, studied a string in tension loaded with weights. He determined that the natural frequency of a system is equal to the square root of its stiffness divided by its mass, \( \omega_n = \sqrt{k/m} \)

**Leonhard Euler**, 1707–1783, was the first person to write down the equation of motion of a harmonically forced, undamped oscillator, 
\[ m\ddot{y} + ky = F \sin \omega t \]. He was also the first person to discover the phenomenon of resonance.
Historical Perspective

Robert Hooke, 1678

\[ F = ky \]

9 years

Isaac Newton, 1687

\[ F = m\ddot{y} \]

63 years

Leonhard Euler, 1750

\[ m\ddot{y} + ky = F \sin \omega t \]
Historical Perspective

Hermann Von Helmholtz, 1821–1894, was the first person to include nonlinearity into the equation of motion for a harmonically forced undamped single degree-of-freedom oscillator. He postulated that the eardrum behaved as an asymmetric oscillator, such that the restoring force was $f = k_1 y + k_2 y^2$ which gave rise to additional harmonics in the response for a tonal input.

John William Strutt, Third Baron Rayleigh, 1842–1919, considered the free vibration of a nonlinear single-degree-of-freedom in which the force-deflection characteristic was symmetrical, given by $f = k_1 y + k_3 y^3$. 
The Duffing Equation

\[ m\ddot{y} + c\dot{y} + k_1y \pm k_3y^3 = F\cos\omega t \]

Georg Duffing 1861-1944
The Duffing Equation: Nonlinear Oscillators and their Behaviour

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Ivana Kovacic and Michael J. Brennan

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Nonlinear stiffness
For a linear system

\[ f = kx \]
Symmetric Nonlinear Spring

Force-deflection

\[ F_s = k_1 y \pm k_3 y^3 \]

Non-dimensional

\[ \tilde{F}_s = \tilde{y} \pm \gamma \tilde{y}^3 \]

\[ \tilde{F}_s = F_s / k_1 l \quad \tilde{y} = y / d_0 \quad \gamma = k_3 d_0^2 / k_1 \]

d_0 = original length of spring
Symmetric Nonlinear Spring

\[ \tilde{F}_s = \tilde{y} + \gamma \tilde{y}^3 \]

\[ \tilde{F}_s = \tilde{y} - \gamma \tilde{y}^3 \]

Note symmetry
Asymmetric Nonlinear Spring

Force-deflection

\[ F_s = k_1 y \pm k_2 y^2 \pm k_3 y^3 \]

Non-dimensional

\[ \tilde{F}_s = \tilde{y} \pm \beta \tilde{y}^2 \pm \gamma \tilde{y}^3 \]

\[ \beta = k_2 d_0 / k_1 \]

\( d_0 = \) original length of spring
Asymmetric Nonlinear Spring

\[ \tilde{F}_s = \tilde{y} - \beta \tilde{y}^2 + \gamma \tilde{y}^3 \]

- \[ \beta = 0 \]

The diagram illustrates the linear and nonlinear behavior of the asymmetric nonlinear spring.
Asymmetric Nonlinear Spring

\[ \tilde{F}_s = \tilde{y} - \beta \tilde{y}^2 + \gamma \tilde{y}^3 \]

\[ \tilde{F}_s = \tilde{y} + \beta \tilde{y}^2 + \gamma \tilde{y}^3 \]

linear

\[ \beta = 0 \]
Asymmetric Nonlinear Spring

\[ \tilde{F}_s = \tilde{y} - \beta \tilde{y}^2 + \gamma \tilde{y}^3 \]

\[ \tilde{F}_s = \tilde{y} + \beta \tilde{y}^2 + \gamma \tilde{y}^3 \]

\[ \tilde{F}_s = \tilde{y} - \beta \tilde{y}^2 - \gamma \tilde{y}^3 \]
Asymmetric Nonlinear Spring

\[
\tilde{F}_s = \tilde{y} - \beta \tilde{y}^2 + \gamma \tilde{y}^3
\]

Note Asymmetry
Pendulum – softening stiffness

Stiffness Moment

\[ ml^2 \frac{d^2 \theta}{dt^2} - mgl \sin \theta = M \cos \omega t \]

\[ \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \]

stiffness moment \( \approx mgl \left( \theta - \frac{\theta^3}{6} \right) \)

Softening
Pendulum – softening stiffness

\[
\frac{k}{k_{\text{lin}}} = 1 - \frac{\theta^2}{2}
\]

\[
\frac{k}{k_{\text{lin}}} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}
\]
Pendulum – softening stiffness

30°  60°  120°  170°
Geometrically nonlinear spring

Force-deflection

\[ F_s = 2ky \left( 1 - \frac{d_0}{\sqrt{y^2 + d^2}} \right) \]

Non-dimensional

\[ F_s = \tilde{y} \left( 1 - \frac{\tilde{d}}{\sqrt{\tilde{y}^2 + 1}} \right), \]

\[ \tilde{F}_s = \frac{F_s}{2kd} \quad \tilde{y} = \frac{y}{d} \quad \tilde{d} = \frac{d_0}{d} \]

\( d_0 = \) original length of spring
Geometrically nonlinear spring

\[ F_s = 2ky \left( 1 - \frac{d_0}{\sqrt{y^2 + d^2}} \right) \]

which approximates to

\[ F_s \approx k_1y + k_3y^3 \]

\[ k_1 = 2k \left( 1 - \frac{d_0}{d} \right) \quad k_3 = k \frac{d_0}{d^3} \]
Geometrically nonlinear spring – non-dimensional

\[ \tilde{F}_s = \tilde{y} \left( 1 - \frac{\tilde{d}}{\sqrt{\tilde{y}^2 + 1}} \right), \]

which approximates to

\[ \tilde{F}_s = \alpha \tilde{y} + \gamma \tilde{y}^3 \]

\[ \alpha = 1 - \tilde{d} \quad \gamma = \frac{\tilde{d}}{2} \]
Geometrically nonlinear spring
– non-dimensional

\[
\tilde{F}_s \quad \tilde{y}
\]

actual

approximation
Geometrically nonlinear spring
– snap through
Geometrically nonlinear spring – snap through

Potential Energy

\[ \tilde{V} = \frac{1}{2} \left(1 - \tilde{d}\right) \tilde{y}^2 + \frac{1}{8} \tilde{d}\tilde{y}^4, \]

\[ \tilde{V} = V/(2kd^2) \]
Example of a snap-through system
Nonlinear Energy Harvesting Device
– snap through
Nonlinear Energy Harvesting Device
Nonlinear Energy Harvesting Device
– snap through

Positive Stiffness
Beam

Negative Stiffness
Magnets
Nonlinear Energy Harvesting Device
– snap through

![Graph showing nonlinear energy harvesting device with force vs. deflection plots representing stiffness, optimal, and soft vertical springs.](image)
Nonlinear Energy Harvesting Device
– snap through

d potential energy
displacement
Vibration Isolation

\[ f_e \]

\[ k_v \quad c \]

\[ m \]

\[ x_t \]

\[ x_e \quad \sim \]
Vibration Isolation

![Diagram of vibration isolation system with labeled components: $k_h$, $k_v$, $c$, $m$, $f_e$, $x_t$, $x_e$, $f_t$, and $l$.]
An Achievable Stiffness Characteristic

![Graph showing normalised stiffness force vs. normalised displacement. The graph highlights a region marked as "Low stiffness".]
Bubble Mount

Static equilibrium position

Very low stiffness (natural frequency)
Large Deflection of Beams
Large deflection of beams

\[ M = EI \left[ \frac{\partial^2 w}{\partial x^2} \right]^3, \]

Causes nonlinear effect

\[ f(x,t) \]

Softening
Large deflection of beams

-in-plane stiffness

\[ \rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \left( T_s + \frac{EA}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 w}{\partial x^2} = f(x,t) \]

Causes nonlinear effect

Hardening
Large deflection of beams
-in-plane stiffness

\[ \rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \left( T_s + \frac{EA}{2l} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 w}{\partial x^2} = f(x,t) \]

Assume simple supports and first mode only

\[ \varphi(x) = \sin \left( \frac{\pi x}{l} \right) \]
\[ w(x,t) = \varphi(x) q(t) \]
\[ f(x,t) = F \delta(x - l/2) \cos \omega t \]

so

\[ m \frac{d^2 q}{dt^2} + k_1 q + k_3 q^3 = F \cos \omega t \]

\[ k_1 = \left[ 1 + T_s l^2 / \left( EI \pi^2 \right) \right] \frac{EI \pi^4}{2l^3} \]
\[ k_3 = \frac{\pi^4 EA}{8l^3} \]
Asymmetrical Systems
Bubble Mount

![Bubble Mount Image]

![Graph Showing Load vs Deflection]

- **LOAD (LBS)**
- **DEFLECTION (IN.)**
- Curves labeled with different numbers (e.g., 50644, 50643, etc.)
Bubble Mount

![Bubble Mount Image]

![Graph showing the relationship between Load (LBS) and Deflection (IN.)]
Asymmetry

stiffness = $k_1 + 3k_3 x^2$
Asymmetry

Symmetric system

\[ F_s = k_1 y + k_3 y^3 \]
Asymmetric system

\[ F_s + F_0 = k_1 y + k_3 y^3 \]
Asymmetry

Asymmetric system

\[ F_s + F_0 = k_1 y + k_3 y^3 \]

\[ F_s = \hat{k}_1 z + \hat{k}_2 z^2 + \hat{k}_3 z^3 \]
Asymmetric System - Cable

\[ m \frac{d^2 q}{dx^2} + k_1 q + k_2 q^2 + k_3 q^3 = F \cos \omega t \]

Mode shapes

Increasing tension

Due to asymmetry - softening effect
Nonlinear damping
### Nonlinear damping

<table>
<thead>
<tr>
<th>DAMPER TYPE</th>
<th>HYSTERESIS LOOP</th>
<th>DAMPING-FORCE TIME HISTORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>VISCOUS</td>
<td><img src="image" alt="Viscous Loop" /></td>
<td><img src="image" alt="Viscous Time History" /></td>
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<tr>
<td>VELOCITY-(n)TH POWER</td>
<td><img src="image" alt="Velocity Loop" /></td>
<td><img src="image" alt="Velocity Time History" /></td>
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<tr>
<td>HYSTERETIC</td>
<td><img src="image" alt="Hysteretic Loop" /></td>
<td><img src="image" alt="Hysteretic Time History" /></td>
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</table>

#### Equations

- **Viscous**
  \[ \xi = C \dot{Z} \]
  \[ \frac{F}{F_0} = 1 - \frac{Z^2}{Z_0^2} \]
  \[ \xi = \xi_0 \cos \omega t, \ \xi_0 = C \xi Z_0 \]

- **Velocity-\(n\)th Power**
  \[ \xi = C_n |\dot{Z}|^n \text{sgn}(\dot{Z}) \]
  \[ \frac{F}{F_0} = C_n^\frac{n-2}{2} \left( 1 - \frac{Z^2}{Z_0^2} \right)^\frac{n}{2} \]
  \[ \xi = \pm \xi_0 \cos^{\frac{1}{n}} \omega t, \ \xi_0 = C_n \xi_0 \frac{Z_0^2}{Z_0^2} \]

- **Hysteretic**
  \[ \xi = C(\omega) \dot{Z} = \frac{C}{\omega} \dot{Z} \]
  \[ \frac{F}{F_0} = 1 - \frac{Z^2}{Z_0^2} \]
  \[ \xi = \xi_0 \cos \omega t, \ \xi_0 = \frac{h}{Z_0} \]
Nonlinear damping

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</tr>
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<td></td>
<td>$\xi = C \dot{z}$</td>
<td>$\xi = E_0 \cos \omega t$, $E_0 = C_0 \omega^2 Z_0^2$</td>
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<tr>
<td>COULOMB</td>
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<td><img src="image" alt="Coulomb damping force" /></td>
</tr>
<tr>
<td></td>
<td>$\xi = \mu F$</td>
<td>$\xi = \xi_0 (Z &gt; 0)$, $\xi = -\xi_0 (Z &lt; 0)$</td>
</tr>
<tr>
<td>QUADRATIC</td>
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Nonlinear damping

Damping force

\[ f_d = \frac{c_h x^2}{(a^2 + x^2)} \dot{x} \]

For \(|x/a| < 0.2\)

\[ f_d = c_h \frac{x^2}{a^2} \dot{x} \]

Damping coefficient is dependent on the square of the amplitude
Summary

• Historical perspective
• Nonlinear stiffness
  • Symmetry
  • Asymmetry
• Nonlinear damping