WAVE PROPAGATION IN ONE-DIMENSIONAL STRUCTURES

Lecture Notes for Day 2

S. V. SOROKIN
Department of Mechanical and Manufacturing Engineering, Aalborg University
1. BASICS OF THE THEORY OF WAVE PROPAGATION IN ELASTIC STRUCTURES (selection)

1.1 Elementary theory of axial deformation of a straight rod
1.2 Time-harmonic behaviour
1.3 General definitions
1.4 Free plane wave of dilatation
1.5 Point excitation
1.6 Green’s function
1.7 Wave interaction (superposition principle)
1.8 Wave reflection from a fixed edge
1.9 Boundary equations of propagation of axial waves
1.10 Wave reflection at the free edge
1.11 Wave transmission through a joint
1.14 Torsion wave in a straight rod of circular or annular cross-section
1.15 Flexural wave – Kirchhoff (Bernoulli-Euler) theory
1.16 Dispersion of flexural waves. Phase velocity. Group velocity
1.17 Forced response. Point excitation. Green’s functions
1.18 Near field and far field
1.23 A simple beam on an elastic foundation. Cut-on frequency

2. ADVANCED TOPICS IN THE THEORY OF WAVE PROPAGATION IN ELASTIC STRUCTURES (selection)

2.1 Flexural wave – Timoshenko theory
2.2 The dispersion equation
2.3 The modal coefficients
2.6 The theory of elasticity – plane stress formulation
2.7 Anti-phase waves predicted by theory of elasticity versus elementary theory of dilatation wave
2.8 In-phase waves predicted by theory of elasticity versus elementary theory of flexural waves and Timoshenko theory
1. BASICS OF THE THEORY OF WAVE PROPAGATION IN ELASTIC STRUCTURES

This Chapter of Lecture Notes covers elementary topics in the theory of wave propagation in elastic rods. However, many phenomena observed in more complicated cases may be identified and explained via analysis of propagation of plane axial, torsion and flexural waves described in the framework of the elementary theory. There are many textbooks on the theory of wave propagation in elastic solids and structures and they differ rather substantially in quality and style. Therefore, this Chapter of Lecture Notes is written in a self-contained way.

This chapter does not contain any graphs, which illustrate parametric studies of obtained analytical solutions. Instead, references to the codes available ‘in house’ are provided in anticipation that during the exercises students run these codes and analyse by themselves roles of parameters.

1.1 Elementary theory of axial deformation of a straight rod

Consider a straight prismatic rod. The elementary theory of its dynamic axial deformation is formulated as follows:

\[
EA \frac{\partial^2 u(x,t)}{\partial x^2} - \rho A \frac{\partial^2 u(x,t)}{\partial t^2} = -q(x,t)
\]  

(1.1)

To derive this equation, it is sufficient to recall its static counterpart (hereafter a reader is referred to J.M. Gere *Mechanics of materials* for all the basics) and apply D’Alembert principle.

This equation has been derived under the ‘plane stress’ assumptions (in this case, in effect, uni-axial stress), which imply that the normal stresses act only in the longitudinal direction and there no constrains on transverse deformation, i.e., \( \sigma_x \neq 0, \sigma_y = \sigma_z = 0 \).

Then \( \sigma_x = E \varepsilon_x = E \frac{\partial u}{\partial x} \). If ‘plane strain’ condition is imposed \( \varepsilon_y = 0 \) (as, for example, in the case of a strip cut from a plate, see Figure 1) along with \( \sigma_z = 0 \), then Hooke’s law reads as \( \sigma_x = \frac{E}{1-\nu^2} \varepsilon_x = \frac{E}{1-\nu^2} \frac{\partial u}{\partial x} \). Since the strip of a unit width is considered, the area of cross-section becomes \( 1 \cdot h \) (\( h \) is the thickness of a plate). Thus, \( EA \) in equation (1.1) is replaced by \( \frac{Eh}{1-\nu^2} \) and \( \rho A \) is replaced by \( \rho h \).
If the rod is bounded at, say, $x = 0$ and $x = L$, this differential equation should be supplemented with boundary conditions (one at each edge) formulated with respect to a linear combination of the displacement and the axial force:

$$X_1 u(x_0, t) + X_2 \frac{\partial u(x, t)}{\partial x} \bigg|_{x=x_0} = 0. \tag{1.2}$$

Furthermore, some initial conditions should be also imposed:

$$u(x, 0) = u_0(x), \quad \frac{\partial u(x, t)}{\partial t} \bigg|_{t=0} = \dot{u}_0(x). \tag{1.3}$$

A solution of the problem (1.1-1.3) describes forced vibrations of the rod.

The eigenvalue problem, which defines a set of natural frequencies of a rod with given boundary conditions, follows from (1-3) as the driving force is set to zero and the initial conditions are disregarded, see, for example, S.S. Rao, *Mechanical Vibrations*. Its solution yields a discrete spectrum of eigenfrequencies and eigenmodes, which present standing waves, see also Section of this chapter.

A problem in wave propagation is formulated for a rod of an infinite length, so that boundary conditions (1.2) are not necessary. They are replaced by so-called radiation conditions at infinity. To understand this concept, it is useful to consider free wave propagation. Free wave $U(x, t)$ is a solution of the homogeneous equation (1.1), rewritten as

$$\frac{\partial^2 U(x, t)}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2 U(x, t)}{\partial t^2} = 0. \tag{1.4}$$

Obviously, this equation is satisfied by an arbitrary function $U(x, t) = \mathcal{U}(x \pm ct), \quad c^2 \equiv \frac{E}{\rho}.$

The actual form of this function is determined by initial conditions (1.3).
1.2 Time-harmonic behaviour

In the previous sub-section, we have considered the wave equation and its solution for an infinitely long rod in the most general form without specification of the excitation conditions. The linear theory of wave propagation in solids and structures, as well as linear acoustics, is concerned with so-called time-harmonic waves. This kind of motion is characterised by simple dependence of displacements, forces, etc., upon time in the form of a trigonometric function, say, \( \cos \omega t \), where \( \omega \) is a circular frequency, measured in radians per second. It is expressed as \( \omega = 2\pi f \) via the frequency \( f \) in Hz.

Such a simplification in no way undermines the generality of analysis, because as is well known, an arbitrary function of time may be presented in the form of Fourier integral, see, for example, S.S. Rao, *Mechanical Vibrations*. In other words, the superposition principle is valid for analysis of linear wave propagation.

Furthermore, analysis of wave propagation is performed in these Lecture Notes under standard assumption that this process has been initiated sufficiently long time ago and the influence of initial conditions may be neglected. In the terminology, adopted in the Theory of Vibrations, the steady state response is analysed.

As has been discussed, a solution of the homogeneous equation (1.4) should be sought in the form

\[
U(x,t) = U_0(x)\cos \omega t \equiv \text{Re}\left[U_0(x)\exp(\pm i\omega t)\right]
\]  

(1.5)

Then the wave equation (4) reduces to the form

\[
\frac{d^2U_0(x)}{dx^2} + \frac{\omega^2}{c^2}U_0(x) = 0
\]

Its solution is \( U_0(x) = A\exp(\hat{k}x) \) and it yields characteristic (or dispersion) equation

\[
\hat{k}^2 + \frac{\omega^2}{c^2} = 0
\]  

(1.6)

The solution is \( \hat{k} = \pm ik \), \( k = \frac{\omega}{c} \).

The general solution of this equation is

\[
U_0(x) = A_1 \exp\left(i\frac{\omega}{c}x\right) + A_2 \exp\left(-i\frac{\omega}{c}x\right)
\]  

(1.7)

Combining each term in formula (1.7) with the time dependence given in (1.5), we obtain four alternative combinations of signs in the formula \( U(x,t) = \text{Re}\left[A\exp(\pm ikx \pm i\omega t)\right] \).

To make a proper choice of these signs, it is necessary to observe that each exponential function of a purely imaginary argument presents a travelling wave of longitudinal deformation of a rod. A choice of the sign in time dependence is arbitrary. In fact, the amount of scientific publications, where this choice is \( \exp(i\omega t) \), is virtually the same at the amount of papers, where \( \exp(-i\omega t) \) is adopted. However, as soon as this choice has been made, the difference between choices of sign in the spatial dependence becomes
obvious. To illustrate it, it is necessary to return to the coupled spatial-temporal formulation of displacement. Let us take $\exp(i\omega)$ so that

$$U(x,t) = A_1 \exp\left(i\frac{\omega}{c_0}x + i\omega t\right) + A_2 \exp\left(-i\frac{\omega}{c_0}x + i\omega t\right) \quad (1.8)$$

Consider the first term in this formula and fix an instant of time $t_0$. Then one obtains a ‘photo’ of the wave profile as sketched in Figure 2 (left). Select a certain (arbitrary) phase, for example,

$$\frac{\omega}{c_0} x_0 + \alpha x_0 = \psi \quad (1.9a)$$

Now consider the next instant of time, $t_0 + \Delta t$ and trace the location of the point $x_0$ in the wave profile with a selected phase

$$\frac{\omega}{c_0} (x_0 + \Delta x) + \omega(t_0 + \Delta t) = \psi \quad (1.9b)$$

Subtraction of (9a) from (9b) yields

$$\Delta x = -c_0 \Delta t \quad (1.9c)$$

This result means that the wave profile (or a phase) moves from right to left. The similar examination of the second term gives the equations
\[ \frac{\omega}{c_0} x_0 - \omega x_0 = \psi . \]  
\[ (1.10a) \]

\[ \frac{\omega}{c_0} (x_0 + \Delta x) - \omega (t_0 + \Delta t) = \psi . \]  
\[ (1.10b) \]

\[ \Delta x = c_0 \Delta t \]  
\[ (1.10c) \]

As is seen in Figure 2 (right), now the wave moves from the left to the right. Furthermore, velocity of wave propagation (it is called the phase velocity) is easily found as

\[ c_{\text{phase}} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = c_0 \]  
\[ (1.11) \]

This velocity is frequency-independent and it is equal to the sound speed.

The alternative formulation of the time dependence is \( \exp(-i \omega t) \) so that

\[ U_-(x, t) = A_1 \exp \left( i \frac{\omega}{c_0} x - i \omega t \right) + A_2 \exp \left( -i \frac{\omega}{c_0} x - i \omega t \right) \]  
\[ (1.12) \]

Apparently, in this case the first term specified a wave with positive phase velocity, i.e., the one travelling from the left to the right, whereas the second term presents a wave, which travels in the opposite direction. The phase velocity has the same direction and the same magnitude as has been found in the previous case.

This discussion has actually introduced the Sommerfeld or the radiation principle, which states that a phase velocity must be directed from a source to infinity.

In these Lecture Notes, the time dependence is always chosen as \( \exp(-i \omega t) \).

### 1.3 General definitions

The wave theory has been developed for applications in virtually all branches of physics and the content of these Lecture notes covers only its very small and simple part. A reader is encouraged to learn more about this fascinating theory, for example, in L.D. Landau, E.M. Livshitz *General course of physics*. For consistency, the basic definitions are briefly presented here following L.I. Slepyan *Models and Phenomena in Fracture Mechanics*:

Complex wave \( \overline{u}(x, y, z, i) = \overline{A}(y, z) \exp(i k x - i \omega t) \)

\[ \overline{u} = \text{Re}[\overline{u}] + i \text{Im}[\overline{u}] \]

\[ \text{Re}[\overline{u}] = \overline{R} \exp[-i \text{Im}(k x) + \text{Im}(\omega t)] \]

\[ \text{Im}[\overline{u}] = \overline{I} \exp[-i \text{Im}(k x) + \text{Im}(\omega t)] \]

\[ \overline{R} = \text{Re}[\overline{A}] \cos[\text{Re}(k x) - \text{Re}(\omega t)] - \text{Im}[\overline{A}] \sin[\text{Re}(k x) - \text{Re}(\omega t)] \]

\[ \overline{I} = \text{Re}[\overline{A}] \sin[\text{Re}(k x) - \text{Re}(\omega t)] + \text{Im}[\overline{A}] \cos[\text{Re}(k x) - \text{Re}(\omega t)] \]

The exponential multiplier is called the envelope.
The trigonometric multiplier is called the carrier.
The carrier wavelength is defined as
\[ \lambda = \frac{2\pi}{\text{Re}[k]} \] (1.14)
The carrier period is defined as
\[ T_0 = \frac{2\pi}{\text{Re}[\omega]} \] (1.15)
The carrier phase velocity is defined as \( \frac{dx}{dt} \) under the condition \( \text{Re}[kx] - \text{Re}[\omega x] = \text{Const} \):
\[ c_{\text{phase}} = \frac{\text{Re}[\omega]}{\text{Re}[k]} \] (1.16)
The group velocity is defined in the same way but for the envelope
\[ c_{\text{group}} = \frac{\text{Im}[\omega]}{\text{Im}[k]} \] (1.17a)
For real frequency and wavenumber the group velocity is defined as
\[ c_{\text{group}} = \frac{d\omega}{dk} \] (1.17b)
and this is in agreement with the previous definition. Indeed, consider the corresponding dispersion relation at a regular point \( k \). An infinitesimal variation of \( k \), \( \delta k = i\varepsilon \), leads to the variation of the frequency, \( \delta\omega = i\varepsilon \frac{d\omega}{dk} \) and to the envelope
\[ \exp \left[ -\varepsilon \left( x - \frac{d\omega}{dk} t \right) \right] \]
propagating with speed \( c_{\text{group}} \). This definition thus relates to a sinusoidal wave with a vanishing envelope.

The name ‘group velocity’ originated from the fact that this is the velocity of a disturbance formed by a group of sinusoidal waves with close frequencies and wavenumbers. In fact, it is the velocity of the non-exponential envelope of these waves. For example,
\[ \sin(kx - \omega x) + \sin[(k + \varepsilon)x - (\omega + \varepsilon c_{\text{group}} x)] = 2\cos \left( \frac{\varepsilon}{2} (x - c_{\text{group}} t) \right) \sin \left[ (k + \varepsilon) x - (\omega + \varepsilon c_{\text{group}}) t \right] - 2\cos \left( \frac{\varepsilon}{2} (x - c_{\text{group}} t) \right) \sin(kx - \omega x) \]
In this expression, the cosine represents the envelope, while the carrier is represented by the sine.

If the phase and group velocities are the same, they are \( k \)-independent and all sinusoidal waves propagate with no change in its size and shape. Otherwise, if the phase and group velocities are different, only a single sinusoidal wave propagates intact, but not a wave packet.
1.4 Free plane wave of dilatation

The free wave travelling from the left to the right is formulated as
\[ U(x,t) = U_0 \exp(ikx - i\omega t) \]  
(1.18)

Then velocity is
\[ v(x,t) = \frac{\partial U(x,t)}{\partial t} = -i \omega U_0 \exp(ikx - i\omega t) \]  
(1.19)

The normal stress in an arbitrary cross-section is
\[ \sigma_x(x,t) = E \frac{\partial^2 U(x,t)}{\partial x^2} = i k E U_0 \exp(ikx - i\omega t) \]  
(1.20)

The instantaneous energy flux density is introduced as (L. Cremer and M. Heckl Structure-Borne sound)
\[ \tilde{E}_{\text{ist}}(x,t) = -\text{Re}[\sigma_x(x,t)]\text{Re}[v(x,t)] = k \omega E U_0^2 \sin^2(kx - \omega t) \]  
(1.21)

It is in common practice to deal with the energy flux averaged over a period of motion,
\[ T = \frac{2\pi}{\omega} . \]

Thus,
\[ \bar{E}(x) = \frac{1}{T} \int_0^T \tilde{E}_{\text{ist}}(x,t) dt = k \omega E U_0^2 \int_0^T \sin^2(kx - \omega t) dt = \frac{1}{2} k \omega E U_0^2 . \]  
(1.22)

Naturally, this quantity remains constant along the length a rod as dictated by the energy conservation law.

This formula may easily be obtained if the following definition is introduced
\[ \bar{E}(x) = -\frac{1}{2} \text{Re}[\sigma_x(x,t)v^*(x,t)] \]  
(1.23)

Since stresses and velocities preserve the same time-dependence and due to the identity \((XY)' = X'Y'\), time- and space-dependence cancels out
\[ \sigma_x(x,t)v^*(x,t) = ik E U_0 \exp(ikx - i\omega t)i \omega U_0 \exp(-ikx + i\omega t) = -k \omega E U_0^2 \]  
(1.24)

Thus, the formula (1.22) is recovered from (1.24)
\[ \bar{E}(x) = \frac{\omega k E U_0^2}{2} = \frac{\rho c \omega^2 U_0^2}{2} . \]  
(1.25)

1.5 Point excitation

Earlier we have considered a wave ‘coming from infinity’ without any discussion of its generation mechanism. Now it is appropriate to specify this mechanism. Let a concentrated force \( F_0 \) be applied as shown in Figure 3 and let us count the axial coordinate from this cross-section.

![Figure 3a](image-url)
The elementary theory of rods is formulated in assumption that all cross-sections of a rod are not distorted in the course of deformation. It is possible if an axial load (see Figure 3b) is produced by normal stresses uniformly distributed in a loaded cross-section, \( F_0 = \sigma_x A \).

\[
\sigma_x = \frac{F_0}{A}
\]

Figure 3b

Due to the symmetry of the wave guide, it is sufficient to consider either left or right semi-infinite rod. Then equation (1.26) becomes homogeneous

\[
\frac{\partial^2 \tilde{U}(x,t)}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2 \tilde{U}(x,t)}{\partial t^2} = 0
\]  

(1.27)

However, the presence of a concentrated force now manifests itself in the boundary condition

\[
\tilde{N}(x,\tau) = -\frac{F_0}{2} \text{sgn}(x) \exp(-i\omega\tau), \quad x = 0 \pm \epsilon, \quad \epsilon \to 0
\]  

(1.28)

The condition (1.28) represents a jump in the force at the point \( x = 0 \). Solution of the problem (1.27-1.28) is sought as

\[
\tilde{U}(x,\tau) = U(x) \exp(-i\omega\tau) = U_0 \exp(ik|x| - i\omega\tau)
\]  

(1.29)

Then the amplitude \( U_0 \) is found from equation

\[
\tilde{N}(x,\tau) = ikEAU_0 \exp(-i\omega\tau) \text{sgn}(x) = -\frac{F_0}{2} \exp(-i\omega\tau) \text{sgn}(x), \quad x = 0 \pm \epsilon, \quad \epsilon \to 0.
\]  

(1.30)

It yields

\[
U_0 = \frac{iF_0}{2EAk}.
\]

The velocity of an arbitrary point (or, that is the same, of an arbitrary cross-section) is

\[
\tilde{V}(x,\tau) = \frac{\omega F_0}{2EAk} \exp(ik|x| - i\omega\tau)
\]  

(1.31)

The force transmitted through an arbitrary cross-section is

\[
\tilde{N}(x,\tau) = -\frac{F_0}{2} \exp(ik|x| - i\omega\tau) \text{sgn}(x)
\]  

(1.32)

The multiplier \( \text{sgn}(x) \) indicates that at any instant of time, the force \( F_0 \) produces deformations of the same magnitude but opposite signs at the same distance from the loaded cross-section to the left and to the right, see Figure 3.

The energy flux in each direction is

\[
\tilde{E}(x) = -\frac{1}{2} \text{Re}[\tilde{N}(x,\tau)\tilde{V}^*(x,\tau)] \text{sgn}(x) = \frac{\omega F_0^2}{8EAk} \text{sgn}(x)
\]  

(1.33)
Naturally, it is positive, when $x > 0$, and it is negative, when $x < 0$, i.e., waves always carry the energy from the source of excitation to infinity. The total energy input is frequency-independent:

$$
F_{\text{input}} = \frac{\omega F_0^2}{4EAk} = \frac{F_0^2}{4\rho c A}.
$$

(1.34)

Input mobility is also frequency-independent:

$$
F_0 \exp(-i\omega t) \frac{2EAk}{\omega} = \frac{2EA}{c} = 2\rho c A.
$$

(1.35)

1.6 Green’s function

The force of unit amplitude $F_0 = 1$ is applied at arbitrary cross-section $x = \xi$. The time dependence $\exp(-i\omega t)$ is omitted hereafter. Then formula (1.29) becomes:

$$
G(x,\xi) = \frac{i}{2EAk} \exp(ik|x-\xi|)
$$

(1.36)

The point, where the force is applied, is usually referred to as a ‘source point’ or an ‘excitation point’. The point, where the deflection is measured is then called a ‘receiver point’ or an ‘observation point’. As is seen, Green’s function depends only upon the distance between a source and a receiver that is obvious from the physical point of view. The reciprocity is readily formulated as $G(x,\xi) = G(\xi,x)$. In what follows, it is very important to keep track of source and observation points, because mixing them up leads to errors in solving problems in wave propagation, transmission and reflection.

1.7 Wave interaction (superposition principle)

The obvious example of wave interaction is provided in the case when two forces act at the rod at the same frequency, but, possibly, with a difference in phases. In the problem of stationary wave propagation, when this process is regarded as initiated infinitely long time ago and any influence of initial conditions is smeared out, one may arbitrarily choose the reference level to count a phase. Let us count the phase from the force acting at the point $\xi_1$ and introduce this force simply as $F_1$. Then the force acting at the point $\xi_2$ may have another amplitude, $F_2$, and its time dependence may be shifted from the force $F_1$ in the angle $\phi$. Thus, the total response of a rod exposed to the excitation by these forces acting simultaneously is

$$
u(x) = F_1 G(x,\xi_1) + F_2 \exp(i\phi) G(x,\xi_2)
$$

$$
= \frac{iF_1}{2EAk} \exp(ik|x-\xi_1|) + \frac{iF_2}{2EAk} \exp(ik|x-\xi_2|).
$$

(1.37)

The axial velocity is

$$
\nu(x) = -i\omega [F_1 G(x,\xi_1) + F_2 \exp(i\phi) G(x,\xi_2)]
$$

$$
= \frac{i\omega F_1}{2EAk} \exp(ik|x-\xi_1|) + \frac{i\omega F_2}{2EAk} \exp(ik|x-\xi_2|).
$$

(1.38)
Its complex conjugate is
\[ v^*(x) = \frac{\omega F_1}{2EAk} \exp(-ik|x - \xi_1|) + \frac{\omega F_2}{2EAk} \exp(-i \varphi) \exp(-ik|x - \xi_2|) \]

Energy input averaged over period is
\[ E_{input} = \frac{1}{2} \text{Re} \left[ F_1 v^*(\xi_1) + F_2 \exp(i \varphi) v^*(\xi_2) \right]. \tag{1.39} \]

Substitution of the expressions
\[ v^*(\xi_1) = \frac{\omega F_1}{2EAk} + \frac{\omega F_2 \exp(-i \varphi)}{2EAk} \exp(-ik|\xi_1 - \xi_2|) \]
\[ v^*(\xi_2) = \frac{\omega F_1}{2EAk} \exp(-ik|\xi_1 - \xi_2|) + \frac{\omega F_2 \exp(-i \varphi)}{2EAk} \]
to (1.39) gives the following formula
\[ E_{input} = \frac{\omega F_1^2}{4EAk} + \frac{\omega F_1 F_2 \cos \varphi}{2EAk} \cos(k|\xi_1 - \xi_2|) + \frac{\omega F_2^2}{4EAk}. \tag{1.40} \]

As is seen from this formula, an addition of the second force may decrease the power input in comparison with the effect of a single load. This result is generic and it constitutes the theoretical background of the active control of sound and vibration.

1.8 Wave reflection from a fixed edge

Consider a semi-infinite rod shown in Figure 4. It is fixed at the origin of coordinates \( u(0) = 0 \) and loaded by the axial force \( F_0 \) at the cross-section \( \xi_1 \).

Superposition principle reads as
\[ u(x) = F_0 G(x, \xi_1) + RG(x,0) = \frac{iF_0}{2EAk} \exp(ik|x - \xi_1|) + \frac{iR}{2EAk} \exp(i k x). \tag{1.41} \]

Due to identity \( G(x, \xi) \equiv G(\xi, x) \) this formula may be also written as
\[ u(x) = F_0 G(\xi_1, x) + RG(0, x) = \frac{iF_0}{2EAk} \exp(ik|x - \xi_1|) + \frac{iR}{2EAk} \exp(i k x) \tag{1.42} \]

In this equation, \( R \) is a reaction force at the fixed edge of the rod. Its magnitude is easily found as \( R = -F_0 \exp(ik\xi_1) \) from the obvious condition \( u(0) = 0 \).

Thus, the total response is formulated as
\[ u(x) = \frac{iF_0}{2EAk} \exp(ik|x - \xi_1|) - \frac{iF_0}{2EAk} \exp(ik + ik \xi_1) \tag{1.43} \]
The amplitude of axial displacement at the excitation point is
\[ u(\xi_1) = \frac{iF_0}{2EAk} - \frac{iF_0}{2EAk} \exp(2ik\xi_1) \] (1.44)
The velocity at this point is
\[ v(\xi_1) = \frac{\omega F_0}{2EAk} - \frac{\omega F_0}{2EAk} \exp(2ik\xi_1) \] (1.45)
Its complex conjugate is
\[ v^*(\xi_1) = \frac{\omega F_0}{2EAk} - \frac{\omega F_0}{2EAk} \exp(-2ik\xi_1) \]
Then the input power becomes
\[ \tilde{E}_{input} = \frac{\omega F_0^2}{4EAk} \left[ 1 - \cos(2k\xi_1) \right]. \] (1.46)

It is worth noting that this quantity is zero, when a fixed edge is located at the distance
\[ \xi_1 = \frac{n\pi}{k} \equiv \frac{n\pi}{\omega}, \ n = 1, 2, 3, \ldots \] This is an example of the mirror anti-sound effect. The reflected wave comes to the excitation point exactly in the anti-phase with the primary source. Apparently, the amplification (doubling) effect is achieved, when \( \xi_1 = \frac{\pi}{2k} + \frac{n\pi}{k} \).

In this case, the reflected wave comes to the excitation point in phase with the primary source.

In this example, it is possible to use elementary considerations to determine the response of a rod. However, it is less obvious how to solve a problem of wave reflection from a free edge and a problem of wave transmission-reflection at the cross-section, where a jump in geometry or material properties of a rod occurs. All these difficulties are conveniently overcome as soon as the concept of boundary equations is introduced.

### 1.9 Boundary equations of propagation of axial waves

Consider an inhomogeneous equation (1.1) with time dependence separated as \( \exp(-i\omega t) \):
\[ \frac{d^2 u(x)}{dx^2} + \left( \frac{\omega}{c_0} \right)^2 u(x) = -\frac{g(x)}{EA} \] (1.47)

It is introduced for a rod located in \( a < x < b \). It should be pointed out that the limit cases \( a \to -\infty \) and \( b \to \infty \) are also captured in the analysis below.

The equation, which involved Green’s function is
\[ \frac{\partial^2 G(x, \xi)}{\partial x^2} + \left( \frac{\omega}{c_0} \right)^2 G(x, \xi) = -\frac{\delta(x-\xi)}{EA} \] (1.48)

It is formulated for the whole ‘space’ \(-\infty < x < \infty\) and it must satisfy the radiation condition. Equation (1.47) is multiplied by Green’s function \( G(x, \xi) \) and integrated over the interval \( a < x < b \). Simultaneously, equation (1.48) is multiplied by the yet unknown
function \( u(x) \) and integrated over the interval \( a < x < b \). Then the latter equation is subtracted from the first one to yield

\[
EA \int_a^b \frac{d^2 u(x)}{dx^2} G(x, \xi) - \frac{\partial^2 G(x, \xi)}{\partial x^2} u(x) \, dx = \int_a^b \left[ -q(x)G(x, \xi) + \delta(x - \xi)u(x) \right] \, dx
\]

(1.49)

Using the fundamental property of \( \delta \)-function, one obtains for any point inside the interval

\[
u(\xi) = EA \int_a^b \frac{d^2 u(x)}{dx^2} G(x, \xi) - \frac{\partial^2 G(x, \xi)}{\partial x^2} u(x) \, dx + \int_a^b q(x)G(x, \xi) \, dx
\]

(1.50)

In this formula and all formulas, which follow from this one, the observation point appears as \( \xi \). Respectively, coordinate \( x \) defines location of the source point.

Integration by parts in the first integral gives

\[
u(\xi) = EA \int_a^b \frac{du(x)}{dx} G(x, \xi) - \frac{\partial G(x, \xi)}{\partial x} u(x) \, dx \bigg|_{x=a}^{x=b} + \int_a^b q(x)G(x, \xi) \, dx
\]

(1.51)

Actually, this is the one-dimensional formulation of Kirchhoff integral, widely used in acoustics. For a given distribution of sources \( q(x) \) within the volume, this formula presents the displacement field inside volume via the boundary values of the displacement potential and its first derivative, which is an axial force.

The case considered in the previous sub-section fits into this formulation, \( q(x) = \delta(x - \xi)F_0 \). Indeed, let us set \( b \to \infty \) and therefore eliminate boundary conditions at the left edge. Furthermore, the condition \( u(0) = 0 \) should be taken into account. Then equation (1.51) is reduced as

\[
u(\xi) = -EA \frac{du(x)}{dx} G(x, \xi) \bigg|_{x=0} + F_0 G(\xi, \xi).
\]

(1.52)

The notation \( R \equiv -EA \frac{du}{dx} \bigg|_{x=0} \) recovers equation (1.42):

\[
u(\xi) = RG(0, \xi) + F_0 G(\xi, \xi) = \frac{iR}{2EAk} \exp(ik\xi) + \frac{IF_0}{2EAk} \exp(i \xi - \xi_0)
\]

(1.53)

If the observation point is placed at the edge of the rod, \( \xi = 0 \), the formula to determine the force is obtained \( R = -F_0 \frac{G(\xi_1, 0)}{G(0,0)} = -F_0 \exp(i k \xi_1) \). Substitution of this formula to (1.53) yields equation (1.43).

### 1.10 Wave reflection at the free edge

As is mentioned, the problem of wave reflection from a free edge, see Figure 5, is conveniently solved by use of this methodology.
In this case the reaction force is zero, \( R \equiv -EA \frac{du}{dx} \bigg|_{x=0} = 0 \) and the equation (1.51) becomes

\[
\begin{align*}
\xi(0) &= \xi(0) = 0, \\
\frac{\partial G(x, \xi)}{\partial x} &= 0, \\
G(0, \xi) &= 0.
\end{align*}
\] (1.54)

This formula contains an unknown amplitude of displacement at the edge \( x = 0 \). To determine this unknown, a boundary equation must be formulated by setting the observation point \( \xi \) at the boundary \( x = 0 \). As follows from equation (2.41), the function \( \frac{\partial G(x, \xi)}{\partial x} \) is not uniquely determined at \( \xi = 0, x = 0 \). Indeed, it is

\[
\frac{\partial G(x, \xi)}{\partial x} = -\frac{1}{2EA} \exp(ik|x-\xi|) \text{sgn}(x-\xi). 
\] (1.55)

To eliminate this ambiguity, one has to observe, that when \( \xi \to 0 \) from inside the rod (see Figure 5), in other words, \( \xi = 0 + \varepsilon, \varepsilon \to 0, \varepsilon > 0 \).

Then \( \frac{\partial G(x, \xi)}{\partial x} \bigg|_{\xi=0+\varepsilon} = -\frac{1}{2EA} \text{sgn}(\varepsilon) = \frac{1}{2EA}. \) The formula (1.54) then yields

\[
u(0) = 2F_0G(\xi, 0).\]

Substitution of this expression to (1.54) gives the solution of the problem we consider,

\[
u(\xi) = \frac{iF_0}{2EAk} \exp(ik\xi + i\xi_1) + \frac{iF_0}{2EAk} \exp(ik|\xi - \xi_1|) 
\] (1.56)

Here it is taken into account that \( \text{sgn}(x-\xi) \equiv -1 \) for an arbitrary cross-section of a rod, \( \xi > x = 0 \).

The amplitude of axial displacement at the excitation point is

\[
u(\xi) = \frac{iF_0}{2EAk} \exp(2ik\xi_1). 
\] (1.57)

The velocity at this point is

\[
u(\xi) = \frac{\omega F_0}{2EAk} + \frac{\omega F_0}{2EAk} \exp(2ik\xi_1). 
\] (1.58)

Its complex conjugate is

\[
u^*(\xi) = \frac{\omega F_0}{2EAk} - \frac{\omega F_0}{2EAk} \exp(-2ik\xi_1). 
\] (1.59)

Then the input power becomes

\[
\tilde{E}_{\text{input}} = \frac{\omega k^2}{4EAk} [1 + \cos(2k\xi_1)]. 
\] (1.60)
As is seen from comparison of equations (1.46) and (1.60), the influence of free edge is virtually the same as the influence of a fixed edge. However, the mirror cancellation and the mirror amplification effects are achieved in the opposite situations regarding the distance between the excitation point and the edge.

Exercise 1.11 – Wave transmission through a joint

The theory

Now consider a more complicated but, perhaps, more realistic case of excitation of a compound rod shown in Figure 6. Its part to the left from \( x = 0 \) has Young’s modulus \( E_1 \), density \( \rho_1 \) and cross-section area \( A_1 \). Respectively, \( E_2 \), \( \rho_2 \) and \( A_2 \) present these parameters of the right semi-infinite element. Let us assume that the axial force is applied at the point \( \xi_1 < 0 \).

\[ F_0 \]

\[ x = \xi_1 \]

\[ x = 0 \]

\[ \text{Figure 6} \]

Unlike cases considered earlier, both the amplitude of force and the amplitude of displacement at the junction cross-section \( x = 0 \) should be determined.

The equation (1.51) for the left segment of a rod, \( -\infty < \xi < 0 \), has the form

\[ u_1(\xi) = E_1 A_1 \left\{ \frac{du_1(x)}{dx} G_1(x, \xi) - \frac{\partial G_1(x, \xi)}{\partial x} u_1(x) \right\}_{x=0} + F_0 G_1(\xi_1, \xi). \]  

(1.61)

Here \( G_1(x, \xi) = \frac{i}{2E_1A_1k_1} \exp(ik_1|x - \xi|) \), \( k_1 = \frac{\omega}{c_1} \), \( c_1 = \frac{E_1}{\sqrt{\rho_1}} \).

The equation (1.51) for the right segment of a rod, \( 0 < \xi < \infty \), is

\[ u_2(\xi) = -E_2 A_2 \left\{ \frac{du_2(x)}{dx} G_2(x, \xi) - \frac{\partial G_2(x, \xi)}{\partial x} u_2(x) \right\}_{x=0}. \] 

(1.62)

Here \( G_2(x, \xi) = \frac{i}{2E_2A_2k_2} \exp(ik_2|x - \xi|) \), \( k_2 = \frac{\omega}{c_2} \), \( c_2 = \frac{E_2}{\sqrt{\rho_2}} \).

The continuity conditions are formulated as

\[ u_1(0) = u_2(0), \] 

(1.63a)

\[ R_1(0) = E_1 A_1 \left. \frac{du_1(x)}{dx} \right|_{x=0} = -R_2(0) = E_2 A_2 \left. \frac{du_2(x)}{dx} \right|_{x=0}. \] 

(1.63b)

The boundary equations are
\[
\frac{1}{2} u_1(0) = R_1(0)G_1(0, 0) + F_0 G_1(\xi_1, 0) \\
\frac{1}{2} u_2(0) = R_2(0)G_2(0, 0)
\]

Thus,
\[
R_1(0) = -R_2(0) = \frac{F_0 G_1(\xi_1, 0)}{G_1(0, 0) + G_2(0, 0)} \quad \text{and} \quad u_1(0) = u_2(0) = 2 \frac{F_0 G_1(\xi_1, 0) G_2(0, 0)}{G_1(0, 0) + G_2(0, 0)}.
\]

The formulas (1.61-1.62) are then written as:

\[-\infty < \xi < 0: \quad u_1(\xi) = R_1(0)G_1(0, \xi) - E_1 A_1 u_1(0) \frac{\partial G_1(x, \xi)}{\partial x} \bigg|_{x=0} + F_0 G_1(\xi_1, \xi) \quad (1.65a)\]

\[0 < \xi < \infty: \quad u_2(\xi) = R_2(0)G_2(0, \xi) + E_2 A_2 u_2(0) \frac{\partial G_2(x, \xi)}{\partial x} \bigg|_{x=0} \quad (1.65b)\]

Implement this theory in a code.

Take the following parameters: \( A_1 = 0.01 \text{m}^2, \quad E_1 = 2.1 \times 10^5 \text{MPa}, \quad \rho_1 = 7800 \text{kg/m}^3, \)
\( A_2 = 0.02 \text{m}^2, \quad E_2 = 2.1 \times 10^5 \text{MPa}, \quad \rho_2 = 7800 \text{kg/m}^3, \quad F_0 = 1 \text{N}, \quad \xi_1 = -4 \text{m}\) and analyse the dependence of power flow \( \tilde{E}(x) = -\frac{1}{2} \text{Re}\left[ \tilde{N}(x, t) \tilde{V}^*(x, t) \right] \) to the left and to the right from the excitation point upon excitation frequency varying from 10 Hz to 10 kHz. Compare these energy fluxes with the input power. Check the energy conservation in spatial coordinate.

### 1.14 Torsion wave in a straight rod of circular or annular cross-section

As is known from Mechanics of Materials, torsion of a straight circular rod is governed by equation similar to the equation of its axial deformation. The dynamic loading is described by equation

\[
GI_p \frac{\partial^2 \tilde{\phi}(x, t)}{\partial x^2} - \rho I_p \frac{\partial^2 \tilde{\phi}(x, t)}{\partial t^2} = -m(x, t) \quad (1.79)
\]

Free time-harmonic (\( \exp(-i\omega t) \)) torsion wave satisfies the equation

\[
GI_p \frac{d^2 \phi(x)}{dx^2} + \rho \omega^2 I_p \phi(x) = 0 \quad (1.80)
\]

Dispersion equation has the form:

\[
\frac{\omega^2}{c_{sh}^2} + \frac{\omega^2}{c_{sh}^2} = 0, \quad c_{sh} = \sqrt{\frac{G}{\rho}} = \frac{c}{\sqrt{2(1+\nu)}} \quad (1.81)
\]

Apparently, all results reported in the previous section are applicable in this case. It is a straightforward exercise to adjust the content of previous Sections to the case of torsion.
1.15 Flexural wave – Kirchhoff (Bernoulli-Euler) theory

Now consider a case of dynamic bending in the framework of elementary theory, which emerges from its static counterpart in the same way as in the previous cases. An arbitrary excitation of flexural deformation is described by equation

\[ EI \frac{\partial^4 \tilde{w}(x,t)}{\partial x^4} + \rho A \frac{\partial^2 \tilde{w}(x,t)}{\partial t^2} = q_w(x,t). \]  
(1.82)

This equation has been derived in ‘plane stress’ assumptions (in this case, in effect, uni-axial stress), which imply that the normal stresses act only in the longitudinal direction and there no constrains on transverse deformation, i.e., \( \sigma_x \neq 0, \sigma_y = \sigma_z = 0 \). Then \( \sigma_x = E \varepsilon_x = Ez \frac{\partial w}{\partial x} \). If ‘plain strain’ condition are imposed \( \varepsilon_y = 0 \) (as, for example, in the case of a strip cut from a plate, see Figure 1) along with \( \sigma_z = 0 \), then Hooke’s law reads as \( \sigma_x = \frac{E}{1-\nu^2} \varepsilon_x = \frac{E}{1-\nu^2} \frac{\partial w}{\partial x} \). Since a strip of unit width is considered, the area of cross-section becomes \( 1 \cdot h \) ( \( h \) is the thickness of a plate). Thus, \( EI \) in equation (1) is replaced by \( \frac{E h^3}{12(1-\nu^2)} \) and \( \rho A \) is replaced by \( \rho h \).

Free time-harmonic (\( \tilde{w}(x,t) = w(x) \exp(-i \omega t) \)) flexural wave is governed by equation

\[ EI \frac{d^4 w(x)}{dx^4} - \rho A \omega^2 w(x) = 0 \]  
(1.82)

Its solution is sought as

\[ w(x) = W \exp(\hat{k}x) \]  
(1.83)

The dispersion equation becomes

\[ EI \hat{k}^4 - \rho A \omega^2 = 0 \]  
(1.84)

Apparently, four roots of this equation give rise to four waves and (1.84) is re-written as

\[ w(x) = W_1 \exp(-kx) + W_2 \exp(-ikx) + W_3 \exp(kx) + W_4 \exp(-ikx), \]  
(1.85)

\[ k = \sqrt{\frac{\omega^2}{c^2}} \sqrt{\frac{A}{I}} = \sqrt{\frac{\omega}{cr_g}}, \quad r_g = \sqrt{\frac{I}{A}} \]

1.16 Dispersion of flexural waves. Phase velocity. Group velocity

As we apply definitions of phase and group velocities we obtain formulas

\[ c_{\text{phase}} = \frac{\omega}{k} = \sqrt{\frac{I}{A}} \sqrt{c \omega} \]  
(1.86a)

\[ c_{\text{group}} = \frac{d \omega}{dk} = 2 \frac{\sqrt{I}}{A} \sqrt{c \omega} = 2c_{\text{phase}} \]  
(1.86b)
Their physical interpretation introduces a concept of dispersion of waves. Its detailed discussion is available in classical texts on physics – for example, Landau and Lifshitz. In very short terms, this is a phenomenon of the frequency dependence of wave propagation speed. It means that an arbitrary (multi-component) signal cannot be conveyed in a flexural wave without distortion. More precisely, the high-frequency components are transported with a higher velocity, than the low-frequency ones.

The elementary theory of flexural deformation is an approximate one and its applicability is limited by a relatively low frequency range. This statement is clearly illustrated by an inspection into the formulas (1.86a,b) as excitation frequency grows, $\omega \to \infty$. Obviously, in this limit both the phase velocity and the group velocity become unbounded. An unbounded growth in phase velocity, strictly speaking, does not violate fundamental laws of physics (this phenomenon will be observed in a number of cases presented in these Notes), because phase velocity is not associated with transportation of mass, momentum of energy. However, an unbounded growth in group velocity means that the energy can be conveyed with an unlimited speed. Naturally, this inconsistency of the elementary theory has been detected and its ranges of validity have been reliably assessed. An improved beam theory suggested by S.P. Timoshenko and the exact solution of the problem of wave propagation in an elastic layer are reported in these Notes. The validity range of elementary and Timoshenko theory is discussed in Chapter 2 of these Notes.

1.17 Forced response. Point excitation. Green’s functions

Similarly to the excitation of a rod by a concentrated axial force, the excitation of a beam by a concentrated transverse force $F_0$ may be considered (see Figure 8):

$$EI \frac{d^4W(x)}{dx^4} - \rho A \omega^2 W(x) = F_0 \delta(x) \quad (1.87)$$

![Figure 8](image)

This excitation is formulated as follows:

$$EI \frac{d^4W(x)}{dx^4} - \rho A \omega^2 W(x) = 0 \quad (1.88a)$$
$$\frac{dW(x)}{dx} = 0, \ x = 0 \pm \varepsilon, \ v \to 0 \quad (1.88b)$$
$$EI \frac{d^3W(x)}{dx^3} = \frac{1}{2} \text{sgn}(x), \ x = 0 \pm \varepsilon, \ \varepsilon \to 0 \quad (1.88c)$$

Solution of the problem (1.88a-c) is
\[ W(x) = \frac{F_0}{4Et^3} \left[ -\exp(-k|x|) + i\exp(ik|x|) \right] \] (1.89)

If the force is applied at the arbitrary cross-section, \( x = \xi \), then
\[ W(x, \xi) = \frac{F_0}{4Et^3} \left[ -\exp(-k|x - \xi|) + i\exp(ik|x - \xi|) \right] \] (1.90)

The slope is
\[ \Gamma(x, \xi) = \frac{F_0}{4kt^3} \left[ \exp(-k|x - \xi|) - \exp(ik|x - \xi|) \right] \operatorname{sgn}(x - \xi) \] (1.91)

The bending moment is
\[ M(x, \xi) = -\frac{F_0}{4k} \left[ \exp(-k|x - \xi|) + i\exp(ik|x - \xi|) \right] \] (1.92)

The shear force is
\[ Q(x, \xi) = \frac{F_0}{4} \left[ \exp(-k|x - \xi|) + i\exp(ik|x - \xi|) \right] \operatorname{sgn}(x - \xi) \] (1.93)

Formulation of the energy flow carried by a bending wave involves both these generalised forces. Respectively, the power produced by them is related to the transverse linear velocity and rotational angular velocity, respectively,
\[ V(x, \xi) = \frac{\omega F_0}{4Et^3} \left[ -\exp(-k|x - \xi|) - \exp(ik|x - \xi|) \right] \] (1.94)

\[ V_1(x, \xi) = \frac{\omega F_0}{4Et^3} \left[ -i\exp(-k|x - \xi|) + i\exp(ik|x - \xi|) \right] \operatorname{sgn}(x - \xi) \]

The amplitude of velocity at the excitation point is complex-valued:
\[ V(0,0) = \frac{\omega F_0}{4Et^3} (1 + i). \]

The total energy input
\[ E_{\text{input}} = \frac{1}{2} \text{Re} [F_0 V^\dagger(0,0)] = \frac{\omega F_0^2}{8Et^3} \] (1.95)

Input mobility appears to be
\[ \frac{F_0}{V(0,0)} = \frac{4Et^3}{\omega(1 + i)} = \frac{4\rho A\omega}{k(1 + i)} \]

Unlike the case of an axial wave, this quantity is complex valued. It means that the force and velocity are not in phase.

The energy flow to the right (or to the left) from an excitation point has two components:
- due to shear force \( E_1(x, \xi) = \frac{1}{2} \text{Re} [Q(x, \xi)V^\dagger(x, \xi)] \), (1.96a)

\[ (\text{here } V^\dagger(x, \xi) = \frac{\omega F_0}{4Et^3} \left[ -i\exp(-k|x - \xi|) + \exp(-ik|x - \xi|) \right]) \]

- due to bending moment \( E_2(x, \xi) = \frac{1}{2} \text{Re} [M(x, \xi)V_1^\dagger(x, \xi)] \) (1.96b)

\[ (\text{here } V_1^\dagger(x, \xi) = \frac{\omega F_0}{4Et^3} \left[ i\exp(-k|x - \xi|) - i\exp(-ik|x - \xi|) \right] \operatorname{sgn}(x - \xi)) \]
It is easy to show that
\[ E_1(x, \xi) = \text{Re} \left[ \exp(-k|x-\xi|) + \exp(ik|x-\xi|) \left( -i \exp(-k|x-\xi|) + \exp(-ik|x-\xi|) \right) \right] \frac{\omega F_0^2}{32EI k^3} \]
\[ = \left( 1 + \exp(-k|x-\xi|) \cos(k|x-\xi|) + \sin(k|x-\xi|) \right) \frac{\omega F_0^2}{32EI k^3} \] (1.97a)
\[ E_2(x, \xi) = \text{Re} \left[ \exp(-k|x-\xi|) + i \exp(ik|x-\xi|) \left( \exp(-k|x-\xi|) - i \exp(-ik|x-\xi|) \right) \right] \frac{\omega F_0^2}{32EI k^3} \]
\[ = \left( 1 - \exp(-k|x-\xi|) \cos(k|x-\xi|) + \sin(k|x-\xi|) \right) \frac{\omega F_0^2}{32EI k^3} \] (1.97b)

Thus, the total power balance is maintained
\[ E_{input} = 2[E_1(x) + E_2(x)] = \frac{\omega F_0^2}{8EI k^3} \] (1.98)

Naturally, as soon as the magnitude of the concentrated force is set to one, we obtain Green’s function:
\[ G(x, \xi) = \frac{1}{4EI k} \left( -\exp(-k|x-\xi|) + i \exp(ik|x-\xi|) \right) \] (1.99)

However, unlike the case of an axial excitation, the loading by a point force does not cover all possible ‘fundamental loading cases’ in the case of flexural waves. An alternative excitation is the one produced by a concentrated moment. As is known, a concentrated moment is modelled as the first derivative of the \( \delta \)-function on a coordinate of the excitation point. To avoid ambiguities, it is necessary to clearly distinguish between the excitation point and the observation point in formulation of the differential equation. It goes as
\[ EI \frac{d^4W_M(x)}{dx^4} - \rho A \omega^2 W_M(x) = M_0 \frac{\partial \delta(x-\xi)}{\partial \xi}. \]
The Green’s function is yielded as \( M_0 = 1 \),
\[ EI \frac{d^4G_M(x, \xi)}{dx^4} - \rho A \omega^2 G_M(x, \xi) = \frac{\partial \delta(x-\xi)}{\partial \xi}. \] (1.100)

Comparison of this equation with (1.87) suggests that the function \( G_M(x, \xi) \) is readily available as the first derivative of the function \( G(x, \xi) \) on co-ordinate \( \xi \),
\[ G_M(x, \xi) = \frac{\partial G(x, \xi)}{\partial \xi} = \frac{1}{4EI k^2} \left[ \exp(-k|x-\xi|) - \exp(ik|x-\xi|) \right] \text{sgn}(\xi-x). \] (1.101)

1.18 Near field and far field

As clearly seen from formulas (1.97a,b), Green’s functions have two components: a travelling wave and an evanescent wave. Although the latter one is present at an arbitrary large distance from the observation point, its contribution becomes ‘invisible’ at a relatively short distance. This point may also be illustrated in inspection into formulas (). The energy distribution between ‘force’ and ‘moment’ components becomes equal sufficiently far from the source. These considerations hint towards the concept of near
and far fields. Apparently, to be in a position to clearly distinguish these zones, one has to select a certain tolerance level, say, for a mismatch of two components of the energy flow.

### 1.23 A simple beam on an elastic foundation. Cut-on frequency

It is often the case that a beam rests on a locally reacting (Winkler) foundation. Then equation of its dynamics reads as \((K\) is the stiffness of the foundation):

\[
EI_y \frac{\partial^4 w(x,t)}{\partial x^4} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + K w(x,t) = q_w(x,t) \tag{1.120}
\]

This simple model permits us to introduce a concept of cut-on frequency, which is very important for understanding wave propagation phenomena as the excitation frequency grows. The standard assumptions that \(q_w(x,t) = 0\) and \(w(x,t) = W \exp(\hat{k}x - i\omega t)\) result in the dispersion equation

\[
EI_y \hat{k}^4 - \rho A \omega^2 + K = 0 \tag{1.121}
\]

Its roots in the low-frequency case, \(\omega < \frac{K}{\rho A}\), are

\[
\hat{k}_{1,2} = \left(\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) \sqrt{\frac{K - \rho A \omega^2}{EI_y}}, \quad \hat{k}_{3,4} = \left(-\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) \sqrt{\frac{K - \rho A \omega^2}{EI_y}}. \tag{1.122}
\]

Then the solution is

\[
w(x) = A_{1-} \exp(\hat{k}_{1}x) + A_{2-} \exp(\hat{k}_{2}x) + A_{1+} \exp(\hat{k}_{3}x) + A_{2+} \exp(\hat{k}_{4}x) \tag{1.123}
\]

Its roots in the high-frequency case, \(\omega > \frac{K}{\rho A}\), are

\[
w(x) = A_{1+} \exp(-kx) + A_{2+} \exp(ikx) + A_{1-} \exp(kx) + A_{2-} \exp(-ikx) \tag{1.124}
\]

\[
k = \sqrt{\frac{\rho A \omega^2 - K}{EI_y}} \tag{1.125}
\]

The difference between these regimes of wave motion becomes obvious, when an excitation by a point force (Green’s function) is considered. In the high-frequency range, the Green’s function has the form (1.99) with a wave number defined by formula (1.125). It is apparent that in the low-frequency range Green’s function consists of two exponentially decaying terms so that the amplitude of displacement vanishes at infinity and the energy flow is identical zero.
2. ADVANCED TOPICS IN THE THEORY OF WAVE PROPAGATION IN ELASTIC STRUCTURES

In this Chapter of Lecture Notes, wave propagation in ‘complex’ elastic wave guides is briefly considered. Firstly, an advanced theory of propagation of flexural and shear waves in a beam is presented. Then the same methodology is applied to study propagation of flexural and shear waves in a three-layered sandwich beam. Furthermore, dispersion curves are analysed for an elastic cylindrical shell. Finally, solution of equations of elasto-dynamics for a layer is obtained to validate elementary theories of wave propagation in a rod (a beam). As a conclusion, the general solution of these equations is discussed.

Naturally, Green’s matrices and boundary equations may be derived to study wave interference phenomena in all considered here wave guides, but this subject lies well beyond the scope of these Lecture Notes.

2.1 Flexural wave – Timoshenko theory

The fundamental difference of this theory from the Kirchhoff theory is that a lateral displacement of the neutral axis of a beam \( w(x,t) \) and an angle of rotation of its cross-section \( \psi(x,t) \) are considered as independent kinematic variables (in the Kirchhoff theory, these two parameters are linked as \( \psi(x,t) = \frac{\partial w(x,t)}{\partial x} \)). Therefore, rotational inertia of cross-sections is taken into account in formulation of the kinetic energy of beam’s motion and the potential energy of shear deformation is also accounted for. The latter task is accomplished by some averaging of shear stresses in a cross-section, which is quantified by so-called shear coefficient \( \kappa \). Detailed derivation and description of Timoshenko beam theory, including discussion of alternative choices of magnitude of this coefficient, is presented in I.H. Shames, C.L. Dym Energy and finite element methods in structural mechanics Taylor and Francis, 1991. Although the Timoshenko theory has been formulated almost one hundred years ago, some of its aspects are still disputed in the research papers. However, as will be shown in Section 2.8, this theory is indeed the best possible approximation to the solution of exact problem in elasto-dynamics.

The set of classical differential equations of motion of a Timoshenko beam

\[
- \rho A \frac{\partial^2 w}{\partial t^2} + \kappa GA \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) + q = 0 \tag{2.1a}
\]

\[
- \rho l \frac{\partial^2 \psi}{\partial t^2} + EI \frac{\partial^2 \psi}{\partial x^2} + \kappa GA \left( \frac{\partial w}{\partial x} - \psi \right) = 0 \tag{2.1b}
\]

As is seen, unlike the elementary beam theory, two independent state variables are introduced here. The first one is the conventional lateral displacement, whereas the
second one presents an angle of rotation of the cross-section. This is a qualitatively new feature of wave propagation, which is discussed in subsequent sections of these Notes.

### 2.2 The dispersion equation

Here, as mathematics becomes slightly more involved, it is practical to use non-dimensional variables. In particular, it is natural to introduce the non-dimensional wave number as \( k = k_c h \) and the frequency parameter as \( \Omega = \frac{\omega h}{c} \), \( c^2 = \frac{E}{\rho} \). The lateral displacement is also scaled by the thickness of a beam \( \bar{W} = \frac{W}{h} \).

\[
\begin{align*}
&w(x,t) = \bar{W} \exp(ik_c x - i\Omega t), \quad \psi(x,t) = \Psi \exp(i k_c x - i \Omega t) \\
&\text{The system of linear algebraic equations, which yields the dispersion equation, is:} \\
&\begin{cases}
-k^2 + \frac{2(1+\nu)}{\kappa} \Omega^2 \bar{W} - ik\Psi = 0, \\
\frac{6ik\kappa}{1+\nu} \bar{W} + \left(-k^2 + \Omega^2 - \frac{6\kappa}{1+\nu}\right) \Psi = 0.
\end{cases}
\end{align*}
\]

Equating to zero the determinant of this system of algebraic equations yields the Timoshenko dispersion equation:

\[
k^4 - \left(1 + \frac{2(1+\nu)}{\kappa}\right) k^2 \Omega^2 - 12\Omega^2 + \frac{2(1+\nu)}{\kappa} \Omega^4 = 0.
\]

This dispersion equation has the same order in wave number as the dispersion equation (1.84). However, it is of the fourth order in frequency parameter \( \left(\frac{2(1+\nu)}{\kappa}\Omega^4\right) \) and it contains a coupling term, \( \left(1 + \frac{2(1+\nu)}{\kappa}\right) k^2 \Omega^2 \). These corrections are asymptotically small at low frequencies, but they play a crucially important role as the frequency grows.

Dispersion curves predicted by this theory for a beam of rectangular cross-section (following Timoshenko, shear coefficient is selected as \( \kappa = \frac{5}{6} \)) are compared with those from elementary theory in Figure 2.1. As is seen, elementary theory describes only one propagating wave (blue curve), which is also detected when Timoshenko theory is used. These two theories agree only in the frequency range of \( \Omega < 0.3 \). However, Timoshenko theory introduces the second propagating wave, which has a cut-on frequency of approximately \( \Omega \approx 1.98 \). As is shown in Section 2.8, this is a wave of dominantly shear deformation.
Figure 2.1 Wave numbers versus excitation frequency. Blue curve – elementary theory, red curves – Timoshenko theory

Figure 2.2 Phase velocities versus excitation frequency. Blue curve – elementary theory, red curves – Timoshenko theory, black horizontal lines – high-frequency asymptotes defining phase velocities of dilatation and shear waves in an unbounded three-dimensional elastic body

The elementary theory predicts an unbounded growth in phase and group velocity of wave propagation as the excitation frequency grows, see formulas (1.86a,b). As discussed
in Section 1.16, it suggests that this theory cannot be applied at sufficiently high frequencies. Similar formulas for the Timoshenko beam are relatively cumbersome and they are not presented here. The dependence of phase and group velocities upon frequency parameter is presented in Figures 2.2 and 2.3. Blue curves are plotted after elementary theory, red curves are plotted after Timoshenko theory. These velocities tend in the high-frequency limit to the velocity of propagation of a wave of dilatation in an unbounded three-dimensional elastic body and to the velocity of propagation of a shear wave in an unbounded three-dimensional elastic body. These asymptotes are plotted as black horizontal lines.

In Figure 2.2, phase velocity is presented. As is seen, elementary theory fails at $\Omega = 0.3$. The validity of Timoshenko theory is assessed in Section 2.8, however, it is clearly seen that it correctly predicts phase velocities of both waves in the limit $\Omega \to \infty$. The second propagating wave has an unbounded phase velocity in vicinity of cut off frequency. As discussed in section 1.16, phase velocity is not associated with transfer of any physical quantity (e.g., mass, momentum or energy) in a wave guide.

![Graph of phase velocity](image)

**Figure 2.3** Group velocities versus excitation frequency. Blue curve – elementary theory, red curves – Timoshenko theory, black horizontal lines – high-frequency asymptotes defining phase velocities of dilatation and shear waves in an unbounded three-dimensional elastic body

In Figure 2.3, group velocity is presented. As discussed, this is a velocity of the energy transportation and it must be bounded. Group velocity given by formula (1.86b) in elementary theory is unbounded and it proves that this theory is invalid at high frequencies. However, the group velocities of both waves predicted by Timoshenko theory tend to their counterparts describing dilatation and shear waves in the theory of elasticity.
It is remarkable that the first propagating wave predicted by Timoshenko theory emerges as a wave of dominantly flexural deformation, but it a high-frequency limit it becomes a wave of pure shear deformation. In opposite, the second propagating wave predicted by Timoshenko theory emerges as a wave of dominantly shear deformation, but it becomes a wave of pure dilatation at high frequencies.

The discussion presented in this subsection omits some details, which are still debated in the literature. The most important are the ‘fine tuning’ of the shear correction factor for the first branch to match the speed of Rayleigh surface wave at high frequencies and the physical interpretation of the second branch at high frequencies.

### 2.3 The modal coefficients

The system of two linear homogeneous algebraic equations (2.3) has infinitely many solutions, when a wave number \( \nu k_j, j=1,2,3,4 \) and a frequency satisfy dispersion equation (2.4). It means that a relation between yet undetermined amplitudes is uniquely defined by any of these algebraic equations. This relation is introduced in the form

\[
\Psi_j = i \beta_j \tilde{W}_j. \tag{2.5}
\]

The modal coefficients \( \beta_j, j=1,2,3,4 \) are defined as (see equations (2.3a,b))

\[
\beta_j = \left( -k_j^2 + \frac{2(1+v)}{\kappa} \Omega^2 \right) \frac{1}{k_j} \equiv \frac{6 \kappa k_j}{1+v} \left( -k_j^2 + \Omega^2 - \frac{6 \kappa}{1+v} \right) \tag{2.6}
\]

The concept of modal coefficients (or modal vectors) has been introduced in analysis of free vibrations of systems with two and more than two degrees of freedom. It may be explained as ‘participation factors’ of independent kinematic components in a given wave.

In Figure 2.4, the modal coefficient (2.6), which corresponds to the first purely propagating wave, versus frequency parameter is plotted by a red line. As is seen, this coefficient specifies the dominantly flexural wave, since the participation of rotation is relatively weak (the modal coefficient is smaller than one). In the case of the elementary beam theory, the link between the amplitudes of the angle of rotation and the lateral displacement is \( \psi(x,t) = \frac{\partial w(x,t)}{\partial x} \), that yields \( \Psi_j = k_j \tilde{W}_j \) and, therefore, \( \beta_j = k_j \) within the elementary theory. Blue line in this Figure presents this curve. As these two curves are compared with the curves in Figure 2.1, it is clear that deviation of predictions of Timoshenko theory from elementary theory occurs simultaneously in the location of dispersion curve of dominantly flexural wave and in the location of its modal coefficient.
In Figure 2.5, the modal coefficient, which corresponds to the second propagating wave, versus frequency parameter is plotted. As is seen, the ratio (2.6) for this wave is much larger than one in the whole frequency range. It suggests that this wave is of dominantly tangential deformation with relatively weak participation of lateral deflection. As
mentioned in the last paragraphs of the previous sub-section, these observations become invalid in the high frequency limit.

2.6 The theory of elasticity – plane stress formulation

The plane stress formulation of time harmonic behaviour of an isotropic elastic layer (see Figure 2.16) in the non-dimensional form reads as:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{1 + \nu}{2} \frac{\partial^2 v}{\partial x \partial y} + \Omega^2 (1 - \nu^2) u = 0 \tag{2.19a}
\]

\[
\frac{1 + \nu}{2} \frac{\partial^2 u}{\partial x \partial y} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} + \Omega^2 (1 - \nu^2) v = 0 \tag{2.19b}
\]

The scaling factor is \( h \) – the thickness of a layer, the frequency parameter is defined as \( \Omega = \frac{\omega h}{c} \), \( c^2 = \frac{E}{\rho} \), time dependence is selected as \( \exp(-i\omega t) \), \( E \) and \( \rho \) are Young’s modulus and density of the isotropic elastic medium.

The fraction-free boundary conditions at the surfaces \( y = \pm \frac{1}{2} \) have the form:

\[
\sigma_y = \frac{E}{\rho} \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} = 0, \tag{2.20a}
\]

\[
\tau_{xy} = \frac{G}{E} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0. \tag{2.20b}
\]

Solution for an infinitely long layer is sought as \( u(x, y) = U(y) \exp(kx) \), \( v(x, y) = V(y) \exp(kx) \). \tag{2.21}

Then equations (2.19) are reduced to the form
\[
\frac{1 - \nu}{2} \frac{d^2 U}{dy^2} + k^2 U + \frac{1 + \nu}{2} k \frac{dV}{dy} + \Omega^2 \left(1 - \nu^2\right) U = 0, \quad (2.22a)
\]
\[
\frac{1 + \nu}{2} k \frac{dU}{dy} + \frac{d^2 V}{dy^2} + \frac{1 - \nu}{2} k^2 V + \Omega^2 \left(1 - \nu^2\right) V = 0. \quad (2.22b)
\]

The solution of problem (2.20-2.22) is sought as:
\[
U(x, y) = \left[A^{(a)}(y) \cosh(\gamma y) + A^{(m)}(y) \sinh(\gamma y)\right] \exp(\kappa x), \quad (2.23a)
\]
\[
V(x, y) = \left[B^{(a)}(y) \sinh(\gamma y) + B^{(m)}(y) \cosh(\gamma y)\right] \exp(\kappa x), \quad (2.23b)
\]

Due to the symmetry in formulation of boundary conditions at the upper and the lower surfaces of an elastic layer, these formulas contain two uncoupled components. The first terms in formulas (2.23a,b) describe ‘anti-phase’ motions of surfaces, the second ones describe their ‘in-phase’ motions. Quite often, the ‘anti-phase’ waves are called symmetric, while ‘in-phase’ waves are called skew-symmetric. The parameters \(\gamma\) are found from the characteristic equation
\[
\left[\frac{1 - \nu}{2} \gamma^2 + k^2 + \Omega^2 \left(1 - \nu^2\right)\right] \gamma^2 + \frac{1 - \nu}{2} k^2 + \Omega^2 \left(1 - \nu^2\right) - \left(\frac{1 + \nu}{2} k\right)^2 = 0. \quad (2.24)
\]

For each root of this equation, \(\gamma_n\), a modal coefficient \(\beta_n \equiv \frac{B_n}{A_n}\) is determined by the relation
\[
\beta_n = -\frac{\frac{1 - \nu}{2} \gamma_n^2 + k^2 + \Omega^2 \left(1 - \nu^2\right)}{\frac{1 + \nu}{2} k\gamma_n} = \frac{\frac{1 + \nu}{2} k\gamma_n}{\gamma_n^2 + \frac{1 - \nu}{2} k^2 + \Omega^2 \left(1 - \nu^2\right)}. \quad (2.25)
\]

Then formulas (2.23a,b) become:
\[
U(x, y) = \left[A^{(a)}(y) \cosh(\gamma_1 y) + A^{(m)}(y) \sinh(\gamma_1 y) + A^{(a)}(y) \sinh(\gamma_2 y) + A^{(m)}(y) \cosh(\gamma_2 y)\right] \exp(\kappa x),
\]
\[
V(x, y) = \left[B^{(a)}(y) \sinh(\gamma_1 y) + B^{(m)}(y) \cosh(\gamma_1 y) + B^{(a)}(y) \cosh(\gamma_2 y) + B^{(m)}(y) \sinh(\gamma_2 y)\right] \exp(\kappa x) \quad (2.26)
\]

The normal stress \(\sigma_y\) and shear stress \(\tau_{xy}\) are formulated as
\[
\sigma_y = \frac{A^{(m)}}{E}(\gamma_1 \beta_1 + \nu k) \sinh(\gamma_1 y) + \frac{A^{(a)}}{E}(\gamma_2 \beta_2 + \nu k) \sinh(\gamma_2 y)
\]
\[
+ \frac{A^{(a)}}{E}(\gamma_1 \beta_1 + \nu k) \cosh(\gamma_1 y) + \frac{A^{(m)}}{E}(\gamma_2 \beta_2 + \nu k) \cosh(\gamma_2 y)
\]
\[
\tau_{xy} = \frac{A^{(a)}}{E}(k \beta_1 + \gamma_1) \cosh(\gamma_1 y) + \frac{A^{(m)}}{E}(k \beta_2 + \gamma_2) \cosh(\gamma_2 y)
\]
\[
+ \frac{A^{(a)}}{E}(k \beta_1 + \gamma_1) \sinh(\gamma_1 y) + \frac{A^{(m)}}{E}(k \beta_2 + \gamma_2) \sinh(\gamma_2 y)
\]

The dispersion equation emerges from four conditions (2.20a,b), which compose a set of four homogeneous linear algebraic equations, as its determinant is equated to zero. These equations are factorized into the set of two uncoupled systems, which describe anti-phase (or symmetric) and in-phase (or skew-symmetric) waves, respectively.

It is expedient to present them as
\[
a_{11}A^{(a)} + a_{12}A^{(m)} = 0, \quad (2.27a)
\]
\[
a_{21}A^{(a)} + a_{22}A^{(m)} = 0, \quad (2.27b)
\]
\[ a_{33}A_1^{(m)} + a_{34}A_2^{(m)} = 0, \]
\[ a_{43}A_1^{(m)} + a_{44}A_2^{(m)} = 0. \]  
(2.27b)

\[ a_{11} = (\beta_1 \gamma_1 + \nu k) \cosh \frac{\gamma_1}{2}, \quad a_{12} = (\beta_2 \gamma_2 + \nu k) \cosh \frac{\gamma_2}{2}, \]
\[ a_{21} = (k \beta_1 + \gamma_1) \sinh \frac{\gamma_1}{2}, \quad a_{22} = (k \beta_2 + \gamma_2) \sinh \frac{\gamma_2}{2}, \]
\[ a_{33} = (\beta_1 \gamma_1 + \nu k) \sinh \frac{\gamma_1}{2}, \quad a_{34} = (\beta_2 \gamma_2 + \nu k) \sinh \frac{\gamma_2}{2}, \]
\[ a_{43} = (k \beta_1 + \gamma_1) \cosh \frac{\gamma_1}{2}, \quad a_{44} = (k \beta_2 + \gamma_2) \cosh \frac{\gamma_2}{2}. \]

The dispersion equation for ‘anti-phase’ waves reads as:
\[ D_{10} = a_{11} a_{22} - a_{12} a_{21} = (\beta_1 \gamma_1 + \nu k)(k \beta_2 + \gamma_2) \cosh \left( \frac{\gamma_1}{2} \right) \sinh \left( \frac{\gamma_2}{2} \right) - 
(\beta_1 \gamma_2 + \nu k)(k \beta_1 + \gamma_1) \cosh \left( \frac{\gamma_1}{2} \right) \sinh \left( \frac{\gamma_2}{2} \right) = 0. \]  
(2.28)

The dispersion equation for ‘in-phase’ waves reads as:
\[ D_{20} = a_{33} a_{44} - a_{34} a_{43} = (\beta_1 \gamma_1 + \nu k)(k \beta_2 + \gamma_2) \sinh \left( \frac{\gamma_1}{2} \right) \cosh \left( \frac{\gamma_2}{2} \right) - 
(\beta_1 \gamma_2 + \nu k)(k \beta_1 + \gamma_1) \sinh \left( \frac{\gamma_1}{2} \right) \cosh \left( \frac{\gamma_2}{2} \right) = 0. \]  
(2.29)

These dispersion equations may be solved by use of various methods, but the computational aspects lie beyond these Lecture Notes.

It is expedient to determine magnitudes of cut-on frequencies by applying the condition \( k = 0 \). Then four sets of cut-on frequencies are readily obtained.

The first one specifies in-phase waves, which cut on as waves of skew-symmetric deformation (specifically, shear waves, \( U_A(y) = A \sin((2n+1)\pi y), V_A(y) \equiv 0 \)). The first wave in this family is the second Timoshenko wave. The cut-on frequencies are
\[ \Omega_{\text{shear},n+1} = \frac{\pi + 2n \pi}{\sqrt{2(1 + \nu)}}, \quad n = 0,1,2,3,\ldots \]  
(2.30a)

The second one specifies in-phase waves, which cut on as waves of symmetric deformation (dilatation waves, \( U_S(y) \equiv 0 \), \( V_S(y) = B_3 \cos(n \pi y) \)). The first wave in this family is a conventional flexural wave. The cut-on frequencies are
\[ \Omega_{\text{dilatation},n+1} = \frac{2n \pi}{\sqrt{1 - \nu^2}}, \quad n = 0,1,2,3,\ldots \]  
(2.30b)

The third one specifies anti-phase waves, which cut on as waves of purely axial symmetric deformation (dilatation waves, \( U_A(y) = A \cos(n \pi y), V_A(y) \equiv 0 \)). The first wave in this family is a conventional axial wave. The cut-on frequencies are
\[ \Omega_{\text{dilation}, n+1}^{\text{cut-on}} = \frac{2n\pi}{\sqrt{2(1+\nu)}} \cdot n = 0, 1, 2, 3, \ldots \] (2.30c)

The fourth one specifies anti-phase waves, which cut on as a wave of purely transverse skew-symmetric deformation (shear waves, \( U_3(y) \equiv 0, \ V_3(y) = B_3 \sin((2n+1)\pi y) \)). The cut-on frequencies are

\[ \Omega_{\text{shear}, n+1}^{\text{cut-on}} = \frac{\pi + 2n\pi}{\sqrt{1-\nu^2}} \cdot n = 0, 1, 2, 3, \ldots \] (2.30d)

It should be pointed out that the condition \( k = 0 \) does not always exactly predict the frequency at which the propagating wave is generated – as is shown in subsequent sections, the mechanism of generation of propagating wave may be different.

2.7 Anti-phase waves predicted by theory of elasticity versus elementary theory of dilatation wave

The elementary theory (see Section 1.1) suggests the following dispersion equation for a plane wave of dilatation in an elastic layer (plane stress):

\[ k^2 - \frac{\rho}{E} \omega^2 = 0 \]

![Figure 2.17 Dispersion curves for anti-phase waves in an elastic layer (magenta) versus the dispersion curve from an elementary rod theory (blue)](image)

The dispersion curve is presented as a blue straight line in Figure 2.17. The dispersion curves obtained by solving equation (2.28) are plotted as red curves. As is seen, the elementary theory matches the first branch (its cut on frequency is zero, see formula (2.30c) for \( n = 0 \)) up to \( \Omega = 2.2 \). This is a relatively high frequency. For example, for a
steel plate of thickness \( h = 2 \text{ cm} \) it is approximately 80 kHz. For a thinner plate, this frequency is even higher. Thus, the elementary theory is sufficiently accurate in the practically meaningful frequency range.

The second wave predicted by general theory of elasto-dynamics cuts on at approximately \( \Omega \approx 3.05 \). The condition (2.30d) gives \( \Omega_{\text{shear,1}}^\text{element} = \frac{\pi}{\sqrt{1 - \nu^2}} = 3.293 \). In fact, this branch is composed by two curves emerging from the point \( \Omega \approx 3.05, \ k \approx 2.02 \) in \((\Omega, k)\) plane. At this point, two branches located in the plane \((\Omega, \text{Im} \ k)\) merge and turn into the growing and decaying branches. The decaying branch is characterised by the fact that its phase velocity is positive, but group velocity is negative. These waves are called negative energy waves. The growing branch merges the solid straight line in this Figure. However, as soon as the third branch emerges, the second one departs from the solid line similarly to the first one. The third wave predicted by general theory of elasto-dynamics cuts on at approximately \( \Omega_{\text{dilatation,1}}^\text{element} = \frac{2\pi}{\sqrt{2(1+\nu)}} \approx 3.897 \), see (2.30c). The fourth branch cuts on at \( \Omega_{\text{dilatation,2}}^\text{element} \approx 7.793 \), see (2.30c).

2.8 In-phase waves predicted by theory of elasticity versus elementary theory of flexural waves and Timoshenko theory

In Figure 2.18, dispersion curves predicted by elementary theory and by Timoshenko theory are compared with the solution of dispersion equation (2.29). First of all, the first branch of Timoshenko dispersion curve perfectly matches the exact solution, whereas the elementary theory has a relatively narrow range of validity. For a steel plate of thickness \( h = 2 \text{ cm} \), this theory is valid up to approximately \( \Omega \approx 0.35 \), which corresponds to 14 kHz.

Timoshenko theory has a much larger range of validity, since its second dispersion curve deviates from the exact solution at around \( \Omega = 3.6 \) or 150 kHz. The excellent agreement between the exact solution and the first Timoshenko curve is actually observed at arbitrary high frequencies. It is readily explained by inspection into frequency-dependence of a modal coefficient for this wave. The rotation component rapidly decays as the frequency grows. Thus, this branch corresponds to the wave of pure shear deformation (sliding of straight vertical cross-sections with respect to each other).
Figure 2.18 Dispersion curves for in-phase waves in an elastic layer (red) versus the dispersion curve from an elementary theory of flexural waves (black) and Timoshenko theory (blue).

The cut-on frequency of the second branch predicted by formula (2.30a), \( \Omega_{\text{cuton},1}^{\text{shear}} = \frac{\pi}{\sqrt{2(1+\nu)}} = 1.948 \) is in very good agreement with Timoshenko theory, which gives \( \Omega_{\text{cuton}} = 1.98 \). In a relatively broad frequency range, the underlying assumption of Timoshenko theory (the linear dependence of axial displacements upon transverse coordinate) agrees with the actual distribution of axial displacements across thickness. The second Timoshenko branch departs from the exact solution as soon as the third wave cuts on, \( \Omega_{\text{cuton},2}^{\text{shear}} = \frac{3\pi}{\sqrt{2(1+\nu)}} = 5.845 \), see formula (2.30a). The fourth wave cuts on as a wave of dilatation, \( \Omega_{\text{cuton},2}^{\text{dilatation}} = \frac{2\pi}{\sqrt{1-\nu^2}} = 6.587 \), see formula (2.30b).

Figure 2.19 presents all dispersion curves predicted by elasto-dynamics and by elementary theories. The anti-phase (red) and in-phase (magenta) curve cross each other, whereas there can be no crossing between curves of the same type of motion.
To conclude this brief analysis of wave propagation in an elastic layer, it should be pointed out that the two-dimensional formulation of the problem employed in Section 2.6 does not permit to capture torsion waves, described for a shaft of a circular or annular cross-section in Section 1.14. However, it is a straightforward exercise to adopt standard assumptions of anti-plane formulation of the problem in elasto-dynamics and therefore to describe propagation of solenoid waves. The first branch of dispersion curves ideally matches the dispersion curve predicted by the elementary theory of torsion.

Therefore, it is possible to conclude that the elementary theories adequately describe waves of all possible types in straight elastic rods.